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Abstract

Full Text

THEORY OF ELASTICITY

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ELASTIC EQUILIBRIUM OF AN ISOTROPIC PLANE RESTING ON A DOUBLY PERIODIC SYSTEM OF POINT SUPPORTS UNDER THE ACTION OF AN ARBITRARY DOUBLY PERIODIC TRANSVERSE LOAD

In works ([1,2]) the solution of the problem of bending of a plate resting on a doubly periodic system of point supports is constructed in double trigonometric series. In work ([3]) the solution of this problem for the case of a uniformly distributed transverse load was obtained in closed form. In the present note a closed-form solution is given for an arbitrary doubly periodic transverse load.

1. Let $\omega_1 = 2,\ \omega_2 = 2le^{i\alpha}$ $(l > 0,\ 0 < \alpha \leq \pi/2)$ be the fundamental periods. The exterior of the system of congruent support points $P = m\omega_1 + n\omega_2$ $(m, n = 0, \pm 1, \pm, \ldots)$ will be called the domain D. The problem consists in constructing Green' s function in D. For this purpose, taking an arbitrary point z_0 in the fundamental parallelogram of periods, we apply to the system of congruent points $P^* = z_0 + m\omega_1 + n\omega_2$ in D a concentrated transverse force s.

The elastic equilibrium of the system is determined by two analytic functions $\Phi(z)$ and $\Psi(z)$ by means of the relations ([4])

$$M_x + M_y = 4D(1+\mu)\operatorname{Re}\Phi(z),$$

$$M_y - M_x + 2iH_{xy} = 2D(1-\mu)\left[\bar{z}\Phi'(z) + \Psi(z)\right],$$

$$N_x - iN_y = -4D\Phi'(z), \qquad \frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y} = \varphi(z) + z\overline{\Phi(z)} + \overline{\psi(z)}, \tag{1}$$

$$w(x,y) = \operatorname{Re}[\bar{z}\varphi(z) + \chi(z)], \qquad \Phi(z) = \varphi'(z), \qquad \Psi(z) = \psi'(z) = \chi''(z),$$

where M_x, M_y and H_{xy} are the bending and twisting moments on the corresponding areas; N_x and N_y are the transverse forces; $w(x,y)$ is the deflection function; $D = Eh^3/12(1-\mu^2)$ is the cylindrical rigidity; h is the thickness of the plate, μ is the Poisson ratio of the material. It is easy to see that all quantities defined in (1) are doubly periodic functions in D.

2. To construct $\Phi(z)$ and $\Psi(z)$, introduce the functions

$$\xi'''(z) = \nu''(z) = \zeta'(z) = -\wp(z), \qquad \gamma_*''(z) = \zeta_*'(z) = -Q(z), \tag{2}$$

where

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n}{}' \left\{ \frac{1}{(z-P)^2} - \frac{1}{P^2} \right\}, \qquad Q(z) = \sum_{m,n}{}' \left\{ \frac{\bar{P}}{(z-P)^2} - 2z\frac{\bar{P}}{P^3} - \frac{\bar{P}}{P^2} \right\},$$

$\wp(z)$ is the Weierstrass elliptic function, and $Q(z)$ is a special meromorphic function.

The properties of the functions (2), indicated in works ([5–7]), make it possible to construct the required analytic functions $\Phi(z)$ and $\Psi(z)$, satisfying all periodicity conditions of the problem.

Put

$$\Phi(z) = A\nu(z) - A\nu(z - z_0) - Aa_0 z + Aa_1,$$

$$\Psi(z) = -A\zeta_*(z) + A\zeta_*(z - z_0) - A\beta\nu(z) + A\beta\nu(z - z_0) - A\beta_0 z + A\beta_1. \tag{3}$$

The constant A in (3) must be determined from the condition that the resultant vector of transverse forces along any closed contour enclosing the origin is equal to s. The remaining constants are determined by the periodicity conditions.

The resultant vector of transverse forces along the curve L has the form (4)

$$Q_z = 2iD[\Phi(z) - \overline{\Phi(z)}]_L. \tag{4}$$

From (4) and (2) we find

$$A = -s/8\pi D. \tag{5}$$

The periodicity condition for the transverse forces is satisfied automatically.

The periodicity condition for the first combination of moments in (1) reduces to the system of equations

$$\operatorname{Re}(\delta_1 z_0 - \alpha_0 \omega_1) = 0, \qquad \operatorname{Re}(\delta_2 z_0 - \alpha_0 \omega_2) = 0, \tag{6}$$

whence, with the aid of Legendre' s relation (8), we find

$$\alpha_0 = \left(\delta_1 - \frac{\pi}{2l\sin\alpha}\right)\frac{z_0}{\omega_1} + \frac{\pi}{2l\sin\alpha}\frac{\overline{z_0}}{\omega_1}. \tag{7}$$

In (6) and (7), $\delta_1 = 2\zeta_1(\omega_1/2)$, $\delta_2 = 2\zeta(\omega_2/2)$, and $\zeta(z)$ is the Weierstrass zeta-function.

The periodicity condition for the second combination of moments in (1) reduces to the system of equations

$$\beta_0\omega_1 + \beta\delta_1 z_0 - \gamma_1 z_0 + \overline{\alpha_0}\omega_1 = 0; \qquad \beta_0\omega_2 + \beta\delta_2 z_0 - \gamma_2 z_0 + \overline{\alpha_0}\omega_2 = 0. \tag{8}$$

The solution of this system has the form

$$\beta = -\frac{\overline{z_0}}{z_0}, \qquad \beta_0 = \left(\gamma_1 - \delta_1 + \frac{\pi}{2l\sin\alpha}\right)\frac{z_0}{\omega_1} + \left(\delta_1 - \frac{\pi}{2l\sin\alpha}\right)\frac{\overline{z_0}}{\omega_1}, \tag{9}$$

where (7)

$$\gamma_1 = 2Q\left(\frac{\omega_1}{2}\right) - \overline{\omega}_1\wp\left(\frac{\omega_1}{2}\right).$$

Finally, the periodicity condition for the complex combination of angles of rotation in (1) leads to the system

$$(\overline{\alpha}_1 + \alpha_1)\omega_1 + \overline{\beta}_1\overline{\omega}_1 = \frac{1}{2}(\delta_1 z_0 + \overline{\delta}_1\overline{z_0})z_0 + \frac{1}{2}\overline{\gamma}_1\overline{z}_0^2,$$

$$(\overline{\alpha}_2 + \alpha_2)\omega_2 + \overline{\beta}_1\overline{\omega}_2 = \frac{1}{2}(\delta_2 z_0 + \overline{\delta}_2\overline{z_0})z_0 + \frac{1}{2}\overline{\gamma}_2\overline{z}_0^2, \tag{10}$$

whence

$$\operatorname{Re}\alpha_1 = \frac{1}{4\omega_1}\left\{\delta_1(z_0^2 + \overline{z}_0^2) - \frac{\pi}{2l\sin\alpha}\left(z_0^2 - z_0\overline{z}_0 + \overline{z}_0^2\right)\right\},$$

$$\overline{\beta}_1 = \frac{1}{2\omega_1}\left\{\delta_1 z_0\overline{z}_0 + (\overline{\gamma}_1 - \overline{\delta}_1)\overline{z}_0^2 + \frac{\pi}{2l\sin\alpha}\left(z_0^2 - z_0\overline{z}_0 + \overline{z}_0^2\right)\right\}. \tag{11}$$

Relations (3), (5), (7), (9), and (11) determine the required Green' s function for the domain D.

3. Let us denote the functions Φ and Ψ corresponding to the concentrated force $s(t)$ applied at the point t by $\Phi[z, t, s(t)]$ and $\Psi[z, t, s(t)]$. We have

$$\Phi[z,t,s(t)] = \frac{s(t)}{8\pi D}\left[\nu(z-t) - \nu(z) + z\alpha_0(t) - \alpha_1(t)\right] = \frac{s(t)}{8\pi D}\Phi^*(z,t),$$

$$\Psi[z,t,s(t)] = \frac{s(t)}{8\pi D}\left[\zeta_*(z) - \zeta_*(z-t) + \beta(t)\nu(z) - \beta(t)\nu(z-t) - \right.$$

$$\left. - z\beta_0(t) - \beta_1(t)\right] = \frac{s(t)}{8\pi D}\Psi^*(z,t). \tag{12}$$

A solution corresponding to an arbitrary doubly periodic transverse load distributed along lines or over areas can be represented through the functions (12) in the usual way

$$\Phi(z) = \frac{1}{8\pi D}\int_\sigma s(t)\Phi^*(z,t)\,d\sigma, \qquad \Psi(z) = \frac{1}{8\pi D}\int_\sigma s(t)\Psi^*(z,t)\,d\sigma. \tag{13}$$

With the aid of (12) and (13) one can solve problems on the bending of a plane supported by columns of arbitrary cross section. The indicated relations can also be used to solve certain boundary-value problems of plate bending.

In conclusion, we note that the solution given here is free of the requirement of symmetry of the periodic system $P = m\omega_1 + n\omega_2$, which occurs in work ([3]).

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Note: Figure translations are in progress. See original paper for figures.