ON NON-SELF-ADJOINT ORDINARY DIFFERENTIAL OPERATORS OF SECOND ORDER

1965

SovietRxiv

Abstract

Full Text

MATHEMATICS

ALLAN M. KRALL

ON NON-SELF-ADJOINT ORDINARY DIFFERENTIAL OPERATORS OF SECOND ORDER

(Presented by Academician L. S. Pontryagin, 16 VII 1965)

For the differential operator $l(y) = -y'' + q(x)y$, considered in $L^2(a, b)$, where a and b are arbitrary, $q(x) = q_1(x) + iq_2(x)$, various results are known (see ([1],[7])). Recently Lidskii ([5]) studied the case when $\lim_{x\to a} q_2(x) = \pm\infty$, $\lim_{x\to b} q_2(x) = \pm\infty$. The non-self-adjoint operator defined by him has a completely continuous resolvent and the spectral expansion is an infinite series. Marchenko ([6]) extended the spectral-expansion formula previously known for self-adjoint operators, but, proceeding in this way, he was forced to leave the space L^2.

We shall consider $l(y) = -y'' + q(x)y$ on (a, b), where $q(x) = q_1(x) + iq_2(x)$ is an arbitrary complex-valued function summable on every subinterval of (a, b) not containing a or b. In addition we assume that $\lim q_2(x) = \gamma$, $\lim q_2(x) = \delta$, $-\infty \le \delta \le \gamma \le \infty$, almost everywhere.

The paper proves the existence of square-summable solutions, defines a non-self-adjoint operator; with the exception of a countable number of eigenvalues, the spectrum of the operator lies on the lines $\nu = \gamma$, $\nu = \delta$, where $\lambda = \mu + i\nu$; the adjoint operator is found, and a spectral expansion is obtained.

1. Solutions of $(l - \lambda)y = 0$ in L^2. Choose an arbitrary point $r \in (a, b)$. Then we have:

Theorem 1.1. *For any $\lambda = \mu + i\nu$ with $\nu \ne \gamma$ there exists a solution $\psi(x, \lambda)$ of the equation $(l - \lambda)y = 0$, belonging to $L^2(r, b)$.*

Choose $s \in (r, b)$ so that $|q_2(x) - \gamma| < |\nu - \gamma|/2$, when γ is finite; if γ is infinite, then for arbitrary $\varepsilon > 0$ choose s so that $|q_2(x) - \nu| > \varepsilon$ for almost all $x \in (s, b)$. The proof then follows from the classical considerations of H. Weyl on the limit point and limit circle in (s, b).

We note that either one or two solutions of $(l - \lambda)y = 0$ belong to $L^2(r, b)$. If two square-summable solutions exist for one point λ, then they both exist for all λ.

Theorem 1.2. *For any $\lambda = \mu + i\nu$ with $\nu \ne \delta$ there exists a solution $\eta(x, \lambda)$ of the equation $(l - \lambda)y = 0$, belonging to $L^2(a, r)$.*

Choose here $t \in (a, r)$ so that $|q_2(x) - \delta| < |\nu - \delta|/2$, if δ is finite; if δ is infinite, then for arbitrary $\varepsilon > 0$ choose t so that $|q_2(x) - \nu| > \varepsilon$ for almost all $x \in (a, t)$, and then argue as above.

2. Solutions of the equation $(l - \lambda)y = f$ **in** L^2**.** In the course of the proof of Theorems 1.1 and 1.2 two functions of λ are found: $M(\lambda)$ and $m(\lambda)$, where $M(\lambda)$ is a limit point or lies in the limit circle associated with b; $m(\lambda)$ is a limit point or lies in the limit circle associated with a.

Theorem 2.1. Let $\nu \neq \gamma$; let s be chosen as in Theorem 1.1; $f \in L^2(r, b)$. Then there exists a solution $y(x, \lambda)$ of the equation $(l - \lambda)y = f$ belonging to $L^2(r, b)$. Let $u = y(s, \lambda)$, $v = y'(s, \lambda)$; then, in the case where only one solution of $(l - \lambda)y = 0$ belongs to $L^2(r, b)$, $y(x, \lambda)$ belongs to $L^2(r, b)$ if and only if

$$v + uM(\lambda) = \int_s^b \Psi(x, \lambda) f(x)\, dx.$$

If both solutions of $(l-\lambda)y = 0$ belong to $L^2(r, b)$, then all solutions of $(l-\lambda)y = f$ belong to $L^2(r, b)$.

Theorem 2.2. Let $\nu \neq \delta$; let t be the same as in Theorem 1.2; $f \in L^2(a, r)$. Then there exists a solution $y(x, \lambda)$ of the equation $(l - \lambda)y = f$ belonging to $L^2(a, r)$. Let $u = y(t, \lambda)$, $v = y'(t, \lambda)$; then, in the case where only one solution of $(l - \lambda)y = 0$ belongs to $L^2(a, r)$, $y(x, \lambda) \in L^2(a, r)$ if and only if

$$v + um(\lambda) = -\int_a^t \eta(x, \lambda) f(x)\, dx.$$

If both solutions of $(l-\lambda)y = 0$ belong to $L^2(a, r)$, then all solutions of $(l-\lambda)y = f$ belong to $L^2(a, r)$.

For differentiable functions f and g, put

$$W[f(x), g(x)] = f(x)g'(x) - f'(x)g(x).$$

If f and g are solutions of $(l - \lambda)y = 0$, then $W[f(x), g(x)]$ does not depend on x and is denoted by $W[f, g]$.

Theorem 2.3. If, for some λ, $\nu \neq \gamma$, $\nu \neq \delta$, one has

$$W[\Psi, \eta] = -\Psi'(t, \lambda) - m(\lambda)\Psi(t, \lambda) = \eta'(s, \lambda) + M(\lambda)\eta(s, \lambda) \neq 0,$$

then for $f \in L^2(a, b)$

$$Rf(x) = \frac{1}{W[\Psi, \eta]} \left[\int_a^x \Psi(x, \lambda)\eta(\xi, \lambda) f(\xi)\, d\xi + \int_x^b \Psi(\xi, \lambda)\eta(x, \lambda) f(\xi)\, d\xi \right]$$

satisfies the equation $(l-\lambda)y = f$ and belongs to $L^2(a,b)$.

$Rf(x)$ satisfies Theorems 2.1 and 2.2. The result follows from Minkowski' s inequality, namely from

$$\|Rf\|_{ab} \le \|Rf\|_{ar} + \|Rf\|_{rb}.$$

For every λ such that $\nu \neq \delta$, $\nu \neq \gamma$, Rf is a bounded operator, whose norm we denote by $C(\lambda)$; $\|Rf\| \le C(\lambda)\|f\|$.

3. The operator L**.** We shall distinguish four cases:

1) Only one solution of $(l-\lambda)y = 0$ belongs to $L^2(a,r)$, and only one belongs to $L^2(r,b)$.
2) Two solutions belong to $L^2(a,r)$, and only one to $L^2(r,b)$.

2′) Only one solution belongs to $L^2(a,r)$, and two to $L^2(r,b)$.

3) Two solutions belong to $L^2(a,r)$, and two to $L^2(r,b)$. We shall consider cases 1), 2), and 3); case 2′) is equivalent to 2).

Let D_0 denote the set of all complex-valued functions f satisfying the conditions:

I. $f \in L^2(a,b)$.

II. f' exists and is absolutely continuous on every subinterval $[\alpha,\beta] \subset (a,b)$, where $a < \alpha < \beta < b$.

III. $lf \in L^2(a,b)$.

Let D be the set of all complex-valued functions f satisfying the conditions:

I. $f \in D_0$.

II.

$$\lim_{x\to a} W[f(x), \Psi(x,\lambda)] = 0, \qquad \lim_{x\to b} W[f(x), \eta(x,\lambda)] = 0$$

for all $\lambda,\ \nu \neq \gamma,\ \nu \neq \delta$.

Define the operator L by $Lf = lf$ for all $f \in D$.

Theorem 3.1. The spectrum of L lies on the lines $\nu = \gamma$, $\nu = \delta$, where $|\gamma| < \infty$ and $|\delta| < \infty$, and at the zeros of the function $W[\Psi,\eta]$. The zeros of $W[\Psi,\eta]$ belong to the point spectrum of the operator L. Finally, Rf, defined in Theorem 2.3, is the resolvent of L.

Theorem 3.2. In case 1), condition II in the definition of D is automatically satisfied for all $f \in D_0$.

In case 2), f automatically satisfies the first half of condition II. If f satisfies the second half at some point λ, where $W[\Psi,\eta]\neq 0$, then the second half of condition II is satisfied for all points λ not lying on $\nu=\gamma, \nu=\delta$.

In case 3), if f satisfies the first half of condition II at the point λ_1 and the second half at the point λ_2 (λ_1 may be equal to λ_2) and $W[\Psi,\eta]=0$ at the points λ_1 and λ_2, then condition II is satisfied for all λ not lying on $\nu=\gamma, \nu=\delta$.

4. The operator adjoint to L. Let E be the set of all complex-valued functions f satisfying the conditions:

I. $f\in D$.

II. $\lim_{x\to a} W[f(x),g(x)]=\lim_{x\to b} W[f(x),g(x)]$ for all $g\in D$.

Theorem 4.1. The domain of definition of L^* is $\overline{E}$, and for all $f\in\overline{E}$

$$L^*f=\overline{L}f.$$

5. Spectral expansion. We shall now assume that $q(x)$ is essentially bounded in every domain not containing a and b, and that the norm $RC(\lambda)$ is uniformly bounded and is of order $O(|\lambda|^{-1/4})$ for large $|\nu|$.

Lemma 5.1. For all $f\in D$, if λ satisfies the conditions $\nu\neq\gamma,\ \nu\neq\delta,\ W[\Psi,\eta]\neq 0$, then

$$Rf=(1/\lambda)(-f(x)+Rg(x)),$$

where $Lf=g$.

Lemma 5.2. $|Rg|=O(C(\lambda)(\lambda^{1/4}))$ as $\lambda\to\infty$, without approaching the lines $\nu=\gamma,\ \nu=\delta$ and the zeros $W[\Psi,\eta]$.

Lemma 5.3. If λ is a simple root of $W[\Psi,\eta]$, then the residue of Rf at the point λ is

$$\int_a^b \Psi(\xi,\lambda)f(\xi)\,d\xi\,\Psi(x,\lambda)\Big/\int_a^b \Psi(\xi,\lambda)^2\,d\xi.$$

We shall assume that all such zeros are simple.

Choose $\varepsilon>0$ so that $\varepsilon<|\gamma-\delta|/2$, when γ and δ are both finite, but otherwise arbitrary. Denote by I the lines from $(-\infty,\gamma+\varepsilon)$ to $(\infty,\gamma+\varepsilon)$, from $(\infty,\gamma-\varepsilon)$ to $(-\infty,\gamma-\varepsilon)$, from $(-\infty,\delta+\varepsilon)$ to $(\infty,\delta+\varepsilon)$, and from $(\infty,\delta-\varepsilon)$ to $(-\infty,\delta-\varepsilon)$.

Theorem 5.4. Let $\{\lambda_i\}_i^\infty$ be the zeros of $W[\Psi,\eta]$, arranged so that $|\lambda_1|\leq|\lambda_2|\leq\cdots$. For any $f\in D$:

$$f(x)=\frac{1}{2\pi i}\int_I Rf(x)\,d\lambda-\sum_i^\infty\int_a^b \Psi(\xi,\lambda_i)f(\xi)\,d\xi\,\Psi(x,\lambda_i)\Big/\int_a^b \Psi(\xi,\lambda_i)^2\,d\xi.$$

Pennsylvania State University
USA

Received
27 V 1965

REFERENCES

1. R. R. D. Kemp, Canad. J. Math., **10**, 447 (1958).
2. Allan M. Krall, Duke Math. J. (in press).
3. Allan M. Krall, Am. Math. Soc. Trans. (in press).
4. Allan M. Krall, Am. Math. Soc. Proc. (in press).
5. B. B. Lidskii, Tr. Moscow Math. Soc., **9**, 45 (1960).
6. V. A. Marchenko, Mat. sbornik, **52** (94), 2, 739 (1960).
7. M. A. Naimark, Tr. Moscow Math. Soc., **3**, 181 (1954).

Note: Figure translations are in progress. See original paper for figures.