# ON THE DENSITY OF FINITE FUNCTIONS AND THE EXTENSION OF CLASSES OF DIFFERENTIABLE FUNCTIONS

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## Abstract

## Full Text

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*MATHEMATICS*

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# ON THE DENSITY OF FINITE FUNCTIONS AND THE EXTENSION OF CLASSES OF DIFFERENTIABLE FUNCTIONS

*(Presented by Academician S. L. Sobolev on 19 IV 1965)*

Let $A$ be a domain of $n$-dimensional Euclidean space. Let $E_n$ be such that from $x \in A$ it follows that $x + y \in A$, if $y = (y_1, \dots, y_n)$, $y_i \geq 0$ $(i = 1, \dots, n)$.

We shall consider the linear manifold of locally summable functions $f(x)$ having generalized derivatives

$$D_i^{k_i} f(x) = \frac{\partial^{k_i}}{\partial x_i^{k_i}} f(x) \qquad (i = 1, \dots, n)$$

with finite seminorm

$$\|f\|_{L_{p,\theta}^{l(\mathbf{m})}(A)} = \sum_{i=1}^{n} \left\{ \int_0^\infty \|\Delta_i^{s_i}(t) D_i^{k_i} f\|_{L_p(A)}^\theta \frac{dt}{t^{1+\theta(l_i - k_i)}} \right\}^{1/\theta}, \tag{1}$$

where $\mathbf{l} = (l_1, \dots, l_n)$, $\mathbf{m} = (m_1, \dots, m_n)$, $m_i = s_i + k_i > l_i > k_i \geq 0$, $1 \leq p \leq \infty$, $1 \leq \theta < \infty$,

$$\Delta_i^s(t)\varphi(x) = \sum_{j=1}^{s} (-1)^{k-j} C_k^j \varphi(x_1, \dots, x_{i-1}, x_i + jt, x_{i+1}, \dots, x_n),$$

$$\|f\|_{L_{p,\infty}^{l(\mathbf{m})}(A)} = \sum_{i=1}^{n} \operatorname*{ess\,sup}_{t>0} t^{-l_i} \|\Delta_i^{s_i}(t) D_i^{k_i} f\|_{L_p(A)}.$$

Declaring functions $f(x)$ and $g(x)$ equivalent if almost everywhere $f(x) - g(x) = P_m(x)$, where $P_m(x)$ is a polynomial of degrees $m_1 - 1, \dots, m_n - 1$, we obtain the linear normed space $\mathcal{L}_{p,\theta}^{l(\mathbf{m})}(\dot{A})$ with norm (1), which is the quotient space by the space of polynomials $P_m(x)$.

We give a representation of functions from $\mathcal{L}_{p,\theta}^{l(\mathbf{m})}(A)$.

Let $\xi(z) \in C^\infty(E_1)$, $\xi(z) = 0$ for $x \le 0$ and $x \ge 1$,

$$\int_0^\infty \xi(z)\,dz = 1,$$

$$\omega_m(z) = \sum_{k=1}^m (-1)^{k+1} C_m^k \frac{1}{k}\xi\Bigl(\frac{z}{k}\Bigr).$$

Hence

$$\int_0^\infty \omega_m(z)\varphi(x+hz)\,dz - \varphi(x) = (-1)^{m+1}\int_0^\infty \xi(z)\Delta^m(hz)\varphi(x)\,dz.$$

Put, for $h_j > 0$ $(j = 1, \dots, n)$,

$$f_{h_1,\dots,h_n}(x) = \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \frac{1}{h_j}\omega_{m_j}\left(\frac{z_j}{h_j}\right) f(x+z)\,dz,$$

$$\tilde{f}_{h_1,\dots,h_n}(x) = \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \frac{1}{h_j^2}\omega_{m_j}\left(\frac{z_j}{h_j}\right)\omega_{m_j}\left(\frac{y_j}{h_j}\right) f(x+z+y)\,dz\,dy, \qquad (2)$$

$$\hat{f}_{h_1,\dots,h_n}(x) = 2f_{h_1,\dots,h_n}(x) - \tilde{f}_{h_1,\dots,h_n}(x).$$

As $\sum_1^n h_i \to 0$, $\hat{f}_{h_1,\dots,h_n}(x) \to f(x)$ in the sense of convergence in $L_{\text{loc}}$ and almost everywhere.

Let $h_i = h^{\sigma_i}$, $\sigma_i > 0$. Then

$$\frac{\partial}{\partial h}\left[\hat{f}_{h^{\sigma_1},\dots,h^{\sigma_n}}(x)\right] = -\sum_{i=1}^n \int_0^\infty \cdots \int_0^\infty h^{-1-\sum_{j=1}{}^n \sigma_j - \sigma_i}\chi_i\left(\frac{y_1}{h^{\sigma_1}},\dots,\frac{y_n}{h^{\sigma_n}}\right)\times\xi\left(\frac{t}{h^{\sigma_i}}\right)\Delta_i^{m_i}(t)f(x+y)\,dt\,dy,$$

where the function $\chi_i(y_1, \dots, y_n) \in C^\infty(E_n)$ is concentrated in the parallelepiped $0 \le y_k \le 2m_k$ $(k = 1, \dots, n)$. Integrating with respect to $h$, for almost all $x$, for any $H > 0$ we obtain the representation

$$f(x) = \hat{f}_{H^{\sigma_1},\dots,H^{\sigma_n}}(x) + \int_0^H \sum_{i=1}^n \int_0^\infty \cdots \int_0^\infty h^{-1-\sum_{j=1}{}^n \sigma_j - \sigma_i}\chi_i\left(\frac{y_1}{h^{\sigma_1}},\dots,\frac{y_n}{h^{\sigma_n}}\right)\times\xi\left(\frac{t}{h^{\sigma_i}}\right)\Delta_i^{m_i}(t)f(x+y)\,dt\,dy\,dh. \qquad (3)$$

For functions $\psi_i(t,x)$, $(t,x) \in (0,\infty) \times A = A^+$, with finite norms

$$\|\psi_i\|_{L_{p,\theta}(A^+)} = \|\|\psi_i\|_{L_p(A)}\|_{L_\theta(0,\infty)}, \qquad \sigma_i = 1/l_i,$$

put

$$g_{(\delta,H)}(x) = \int_\delta^H \sum_{i=1}^n \int_0^\infty \cdots \int_0^\infty h^{-1-\sum_{j=1}{}^n \sigma_j - \sigma_i} \chi_i \left(\frac{y_1}{h^{\sigma_1}}, \dots, \frac{y_n}{h^{\sigma_n}}\right) \times \xi \left(\frac{t}{h^{\sigma_i}}\right) t^{1/\theta + l_i} \psi_i(t, x+y)\, dt\, dy\, dh. \tag{4}$$

With the aid of Hardy' s and Minkowski' s inequalities, after the corresponding changes of variables one can show that, for $1 \le p \le \infty$, $1 \le \theta \le \infty$,

$$\|g_{(\delta,H)}\|_{\mathcal{L}^{1(\mathbf{m})}_{p,\theta}(A)} \le C \sum_{i=1}^n \int_0^1 u^{l_i} \left\{ \int_{u\delta_i}^{uH^{\sigma_i}} \|\psi_i\|^\theta_{L_p(A)}\, dt \right\}^{1/\theta} du \le C \sum_{i=1}^n \|\psi_i\|_{L_{p,\theta}(A^+)}. \tag{5}$$

**Corollary.** Since

$$\sum_{i=1}^n \left\| t^{-1/\theta - l_i} \Delta_i^{m_i}(t) f(x) \right\|_{L_{p,\theta}(A^+)} \le \|f\|_{\mathcal{L}^{1(\mathbf{m})}_{p,\theta}(A)},$$

it follows from (3), (4), (5) that, for $\sigma_i = 1/l_i$, $H \to 0$,

$$\left\| f - \hat{f}_{H^{\sigma_1}, \dots, H^{\sigma_n}} \right\|_{\mathcal{L}^{1(\mathbf{m})}_{p,\theta}(A)} \to 0,$$

in any case when $1 \le \theta < \infty$, and for those functions from $\mathcal{L}^{1(\mathbf{m})}_{p,\infty}(A)$ for which

$$\|\Delta_i^{m_i}(t) f\|_{L_p(A)} = o(t^{l_i}) \quad \text{as } t \to 0 \quad (i = 1, \dots, n),$$

i.e. the density of the set $C^\infty(\bar{A})$ in these classes.

For $H = \infty$, the right-hand side of (4) is understood as an integral convergent in itself in $\mathcal{L}^{1(\mathbf{m})}_{p,\theta}(A)$ as $H \to \infty$, which holds, in any case, for $1 \le \theta < \infty$, and for $\theta = \infty$ for those functions $\psi_i(t,x)$ for which

$$\|\psi_i\|_{L_p(A)} = o(1) \quad \text{as } t \to \infty.$$

Let

$$\varphi^{(i)}(t,x) = t^{-1/\theta - l_i + k_i} \Delta_i^{s_i}(t) D_i^{k_i} f(x).$$

As $\sum_1^n h_i \to \infty$

$$\|f_{h_1,\dots,h_n}\|_{L^{1(\mathbf{m})}_{p,\theta}(A)} = \sum_{i=1}^{n} \|\varphi^{(i)}_{h_1,\dots,h_n}\|_{L_{p,\theta}(A+)} \to 0, \qquad \|\hat{f}_{h_1,\dots,h_n}\|_{L^{1(\mathbf{m})}_{p,\theta}(A)} \to 0 \quad (6)$$

at least for $1 < p \leqslant \infty^*$, $1 \leqslant \theta \leqslant \infty$, and under the following additional requirements:

a) for $p = \infty$,

$$\left\{ \int_\alpha^\beta |\varphi^{(i)}(t,x)|^\theta \, dt \right\}^{1/\theta} \to 0 \quad (x \in A,\ |x| \to \infty),$$

for which it is sufficient that

$$\sum_1^n \left| D_i^{k_i}(f - P_m(f;x)) \right| \to 0 \quad (x \in A,\ |x| \to \infty);$$

b) for $\theta = \infty$,

$$\|\varphi^{(i)}(t,x)\|_{L_p(A)} \to 0 \quad (t \to 0,\ t \to \infty),$$

$$\operatorname*{ess\,sup}_{\alpha<t<\beta} \sum_{i=1}^n \|\varphi^{(i)}(t,x)\|_{L_p(x \in A,\ |x|>R)} \to 0 \quad (R \to \infty)$$

for every $[\alpha, \beta] \subset (0, \infty)$. The last condition, for $1 < p < \infty$, is fulfilled if

$$\sum_{i=1}^n \left\| D_i^{k_i}(f - P_m(f;x)) \right\|_{L_p(A)} < \infty.$$

In the proof one uses the approximation of the functions $\varphi^{(i)}(t,x)$ in $L_{p,\theta}(A+)$ by finite and bounded ones.

When (6) holds, from (3) we obtain the representation in $\mathcal{L}^{1(\mathbf{m})}_{p,\theta}(A)^{**}$

$$f(x) = \int_0^\infty \sum_{i=1}^n \int_0^\infty \cdots \int_0^\infty h^{-1-\sum_{j=1}^{n}\sigma_j - \sigma_i} \chi_i \left( \frac{y_1}{h^{\sigma_1}}, \dots, \frac{y_n}{h^{\sigma_n}} \right) \times$$

$$\times \xi \left( \frac{t}{h^{\sigma_i}} \right) \Delta_i^{m_i}(t) f(x+y) \, dt \, dy \, dh. \quad (7)$$

For finite $f(x) \in C^\infty(\bar{A})$, (3) and (7) hold everywhere in $A$ and give representations analogous to the representations of V. P. Il' in.

**Theorem 1.** The functions $f(x)$ of the space $\mathcal{L}^{1(\mathbf{m})}_{p,\theta}(A)$, $1 < p \leqslant \infty$, $1 \leqslant \theta \leqslant \infty$ (for $p = \infty$ or $\theta = \infty$ satisfying the additional conditions a), b)) can be approximated with arbitrary accuracy by infinitely differentiable finite functions.

For the proof it is enough to observe that any such function $f(x)$ is approximated with arbitrary accuracy by the functions $(0 < \delta < H < \infty)$

$$f_{(\delta,H)}(x) = \int_\delta^H \sum_{i=1}^n \int_0^\infty \cdots \int_0^\infty h^{-1-\sum_{j=1}{}^n \sigma_j - \sigma_i} \chi_i\left(\frac{y_1}{h^{\sigma_1}}, \dots, \frac{y_n}{h^{\sigma_n}}\right) \times$$
$$\times \xi\left(\frac{t}{h^{\sigma_i}}\right) \Delta_i^{m_i}(t) f(x+y)\, dt\, dy\, dh,$$

and the function $f_{(\delta,H)}(x)$ (for fixed $\delta > 0,\ H < 0$) by functions $g_{(\delta,H)}(x)$ with finite (and infinitely differentiable) functions $\psi_i$ with compact support in $A+$ (approximating $t^{-1/\theta - l_i}\Delta_i^{m_i}(t)f(x)$). But such functions $g_{(\delta,H)}(x) \in C^\infty(\bar{A})$ are finite, as was required to prove.

**Theorem 2.** The space $\mathcal{L}_{p,\theta}^{1(\mathbf{m})}(A)$, $1 \leqslant p \leqslant \infty$, $1 \leqslant \theta \leqslant \infty$, is complete.

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* The same also holds for $p = 1$ for a domain $A$ which, for some $i$, contains entirely no straight line $x_k = x_k^0$ $(1 \leqslant k \leqslant n,\ k \neq i)$ and $h_i \to \infty$.

** The right-hand side of (7), for every $R < \infty$, converges in $L_p(|x| < R,\ x \in A)$ to $f(x) - P(f, x)$.

The proof is carried out by projecting $\mathcal{L}_{p,\theta}^{1(m)}(A)$ onto the space of polynomials $P_m(x)$.

**Remark.** Let $\widetilde{\mathcal{L}}_{p,\theta}^{1(m)}(A)$ denote the closure of finite functions $f(x) \in C^\infty(\overline{A})$ in the norm $\mathcal{L}_{p,\theta}^{1(m)}(A)$. Such spaces were studied in ([2]). For $1 < p < \infty$, $1 \le \theta < \infty$, $\widetilde{\mathcal{L}}_{p,\theta}^{1(m)}(A)$ coincides with $\mathcal{L}_{p,\theta}^{1(m)}(A)$. From estimate (5) it follows that, for $1 \le p < \infty$, $1 \le \theta < \infty$, $\widetilde{\mathcal{L}}_{p,\theta}^{1(m)}(A)$ coincides with the functions $g_{(0,\infty)}(x)$ representable in the form (4) through functions

$$\psi_i(t, x), \quad \sum_{i=1}^n \|\psi_i\|_{L_{p,\theta}(A^+)} < \infty.$$

The question of the coincidence of $\widetilde{\mathcal{L}}_{1,\theta}^{1(m)}(E_n)$ and $\mathcal{L}_{1,\theta}^{1(m)}(E_n)$ remains open.

**Theorem 3.** *Let*

$$s_i + k_i \ge m_i > l_i > k_i \ge 0 \quad (i = 1, \dots, n).$$

*In the space $\mathcal{L}_{p,\theta}^{1(m)}(A)$, $1 < p < \infty$, $1 \le \theta < \infty$, all norms of the form (1) are equivalent for different $s_i, k_i$, and on finite functions $f(x) \in C^\infty(\overline{A})$ $(1 \le p \le \infty,\ 1 \le \theta \le \infty)$–for different $s_i, k_i, m_i$.*

The second part of the theorem is known ([3]). For the proof it is convenient to use representation (7) and the same estimates as in the derivation of (5).

**Theorem 4.** *The spaces* $\widetilde{\mathcal{L}}^{1(m)}_{p,\theta}(A)$ $(1 \le p \le \infty,\ 1 \le \theta \le \infty)$ *admit a linear bounded extension from* $A$ *to* $E_n$.

The proof is carried out by extending the functions

$$\varphi_i(t,x) = \Delta_i^{m_i}(t) f(x)$$

in (7) by zero for $x \in E_n \setminus A$ and using estimates (5). The idea of this method belongs to Calderon ([4]); for $l_1 = \cdots = l_n$ an analogous result was obtained in ([5]); the possibility of applying the method in the present situation was also suggested by V. P. Il' in.

An analogous theorem is also valid for the spaces $B^l_{p,\theta}(A)$ (introduced in (6)) with the normalization

$$\|f\|_{B^l_{p,\theta}(A)} = \|f\|_{L_p(A)} + \|f\|_{{\mathcal{L}_{p,\theta}}^{1(m)}(A)}.$$

The possibility of extending $\widetilde{\mathcal{L}}^{1(m)}_{p,\theta}(A)$ for

$$A = E_m \times (0,\infty)^{n-m}$$

was established by V. A. Solonnikov by the method of Whitney and Hestenes ([1]).

Let, for a domain $\Omega$, $U_\delta$ denote the intersection of the $n$-dimensional $\delta$-neighborhood of the set $U$ with $\Omega$. Suppose that for $\Omega$ there exists a finite covering $\{U^{(k)}\}_{k=0}^K$ with the properties:
1) $\Omega \subset \bigcup_{k=0}^K U^k \subset \Omega$,
2) there exist cones

$$V_k\{(x, Q_k) > |x| \cos\gamma,\ |x| < r\}$$

such that, for $x \in U^{(k)}_{2\delta}$, $x + V_k \subset \Omega$, and for

$$x \in \partial\Omega \cap \overline{U}^{(k)}_{2\delta}$$

one has $x - V_k \subset E_n \setminus \Omega$. These conditions are satisfied, for example, by a bounded domain $\Omega$ with an $(n-1)$-dimensional boundary $\partial\Omega$ locally satisfying the Lipschitz condition. Put

$$\|f\|_{B^l_{p,\theta}(\Omega)} = \sum_{|\alpha| \le \bar{l}} \|D^\alpha f\|_{L_p(\Omega)} + \sum_{|\alpha| \le \bar{l}} \left\{ \int_{E_n} \|\Delta^2(t) D^\alpha f\|^\theta_{L_p(\Omega)^{(2t)}} \frac{dt}{|t|^{m+\theta\lambda}} \right\}^{1/\theta},$$

where $\bar{l}$ is an integer; $0 < l - \bar{l} = \lambda \le 1$; $\Omega^{(t)}$ is the largest domain for which

$$\bigcap_{0 \le \rho \le 1} (\Omega^{(t)} + \rho t) \subset \Omega.$$

For $\lambda < 1$, $\Delta^2(t)$, $\Omega^{(2t)}$ may be replaced by $\Delta(t)$, $\Omega^{(t)}$.

**Theorem 5.** *The spaces $B^l_{p,\theta}(\Omega)$ admit a linear bounded extension of functions from $\Omega$ to $E_n$. The spaces $B^l_{p,\theta}(A)$ and $B^l_{p,\theta}(A)$ coincide for $l = (l, \dots, l)$; their norms are equivalent.*

*Correction note.* After the paper had been submitted to the editorial office, it became known to us that the theorem on the extension of functions from the space $B^l_{p,\theta}$ had also been proved by V. P. Il' in.

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*Note: Figure translations are in progress. See original paper for figures.*