ASYMPTOTICS OF SOLUTIONS OF LINEAR DYNAMIC MODELS OF ECONOMIC SYSTEMS WITH DISCRETE TIME

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Abstract

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ASYMPTOTICS OF SOLUTIONS OF LINEAR DYNAMIC MODELS OF ECONOMIC SYSTEMS WITH DISCRETE TIME

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Questions concerning the asymptotic behavior of trajectories of linear closed models of an expanding economy have recently attracted much attention from mathematical economists (see, for example, ([1]-[3])). In ([1], [2]) and other works, so-called turnpike theorems in weak form are established for various classes of closed models. From Theorem 4 of the present work there follows a turnpike theorem in weak form for a broad class of closed models (in a certain sense the broadest, by virtue of Theorem 5). Of particular interest from the practical point of view is the problem of determining the asymptotics in the most general linear model—the dynamic problem of linear programming (l.p.) (for the formulation of this problem see ([4])). Theorems 1 and 2 establish a connection between the dynamic problem of l.p. and a closed linear model of an expanding economy.

I, 1. The concepts of an ingredient, a technological method, and a linear programming problem (l.p.) are assumed to be known. The corresponding definitions may be found in ([5]).

I, 2. A dynamic l.p. problem (D) is a l.p. problem

$$\sum_s a_i^{(s)} h^{(s)} \geqslant b_i, \qquad i = 1, \dots, n;$$

$$\sum_s a_0^{(s)} h^{(s)} \geqslant \max, \qquad s = 1, \dots, S,$$

possessing the following properties:

a) the set of indices of ingredients $\{i\}$ $(i = 1, \dots, n)$ is decomposed into non-intersecting subsets $\{i_1\}, \dots, \{i_t\}, \dots, \{i_T\}$, ordered by increasing t $(t = 1, \dots, T)$;

b) among the methods $a^{(s)}$ there is none such that $a_{i_t}^{(s)} > 0$, $a_{i_{t'}}^{(s)} < 0$, and $t' > t$ (irreversibility of time);

c) the index of the maximized ingredient belongs to the set $\{i_T\}$;

d) $b_{i_1} \leqslant 0,\ i_1 \in \{i_1\}$.

I, 3. With the dynamic l.p. problem D there are associated the sets $R_t(D)$.

$$R_t(D) = \{X \mid x_{i_{t+1}} = \sum_{s_t} a_{i_{t+1}}^{(s_t)} h^{(s_t)} - b_{i_t};\ \sum_s a_i^{(s)} h^{(s)} \geqslant b_i\}.$$

Here the index s_t ranges over the indices of methods $a^{(s)}$ for which all $a_{i_{t+1}}^{(s)} \geqslant 0$.

I, 4. An object defined by a sequence of cones $\{Z_t\}$ and a sequence of vectors $\{f(t)\}$, $t = 1, 2, \ldots$, is called a model of Gale type $\mathfrak{M}$. The cone Z_t is convex and closed, situated in Euclidean space of $n(t) + n(t+1)$ dimensions. A point $(x, y) = (x_1, \ldots, x_{n(t)}, y_1, \ldots, y_{n(t+1)})$ belonging to Z_t is called a techno-

logical process, and X is called the vector of costs, Y the vector of output. The vector $f(t) = (f_1(t), \ldots, f_{n(t+1)}(t))$ is called the load vector. The state of the model $\mathfrak{M}$ in period t is the nonnegative vector $X(t) = (x_1(t), \ldots, x_{n(t)}(t))$. In substantive terms, $X(t)$ shows the quantity of each product available at the beginning of period t. X_0 is the initial state of the model, $X_0 = X(1)$. The sequence of states $\{X(t)\}_{t=1}$ is called admissible for the initial state $X_0 = X(1)$ if the process $(X(t), X(t+1)) \in Z_t$ for all t $(t = 1, \ldots, T-1)$.

I, 5. Additional definitions.

$$R_{X_0,1} = \{Y' \mid (X_0, Y) \in Z_1,\ Y' = Y + f(1)\},$$

$$R_{X_0,t} = \{Y' \mid (X, Y) \in Z_t,\ X \in R_{X_0,t-1},\ Y' = Y + f(t)\}.$$

The set $R_{X,t}(i_1, \ldots, i_k)$ is the projection of the set $R_{X,t}$ onto the subspace of products $i_1, \ldots, i_k$.

I, 6. Let the products $i_1, i_2, \ldots, i_{k(t)}$ participate in the models $\mathfrak{M}(Z_t, f(t))$ and $\mathfrak{M}'(Z_t', f'(t))$. We shall call $\mathfrak{M}$ and $\mathfrak{M}'$ equivalent with respect to the products $i_1, \ldots, i_{k(t)}$ if: 1) for any initial state of the model $\mathfrak{M}$, $X_0 = (\ldots, x_{i_1}, \ldots, x_{i_{k(1)}}, \ldots)$, one can choose an initial state of the model $\mathfrak{M}'$, $X_0' = (\ldots, x_{i_1}, \ldots, x_{i_{k(1)}}, \ldots)$, and 2) for any initial state of the model $\mathfrak{M}'$, $X_0' = (\ldots, x_{i_1}, \ldots, x_{i_{k(1)}}, \ldots)$, one can choose an initial state of the model $\mathfrak{M}$, $X_0 = (\ldots, x_{i_1}, \ldots, x_{i_{k(1)}}, \ldots)$, such that the sets $R_{X_0,t}(i_1, \ldots, i_{k(t)})$ and $R_{X_0',t}(i_1, \ldots, i_{k(t)})$ coincide. The model $\mathfrak{M}(Z_t, f(t))$ is called a model of Neumann type if the cones Z_t are polyhedral, i.e., are defined by a finite number of generators (the so-called basic methods). Denote by $k(t)$ the number of ingredients of problem D belonging to the set $\{i_t\}$.

Theorem 1. For any dynamic problem of linear programming D one can construct a model of Neumann type $\mathfrak{M}(Z_t, f(t))$ such that the sets $R_t(D)$ and $R_{X_0,t}(i_1, \ldots, i_{k(t)})$ coincide for all t, $t = 1, 2, \ldots$.

Theorem 2. If in a model of Neumann type $\mathfrak{M}(Z_t, f(t))$, beginning with some period t_0: a) the cone Z_t does not change, b) $f(t) = \sum_j c_j g_j(t)$, where $c_j = (c_j^1, \dots, c_j^{n(t_0)})$,

$$g_j(t) = \sum_{\tau=1}^{\theta_j} \lambda_\tau^j (t - \tau), \qquad j = 1, \dots, p,$$

then one can construct an equivalent (with respect to the products $1, \dots, n(t_0)$) closed (i.e. $f(t) \equiv 0$) model $\mathfrak{M}'(Z')$, in which the cone Z does not change over time.

II, 1. Extremal problems associated with a model of Neumann type $\mathfrak{M}$.

a) Given an initial state X_0, a nonnegative vector of values C, and the length T of the planning period. Determine an admissible, for X_0, sequence $\{X(t)\}_{t=1}^{t=T+1}$ such that the value $(X(T)C)$ assumes its maximum value (the definition of the maximum of (C, X) on the set $R_{X_0,T-1}$). Denote the solution of this problem by $\{X_T(t)\}_{t=1}^{t=T-1}$.

b) Given an initial state X_0. Find an admissible, for X_0, trajectory of states $\overline{X} = \{\overline{X}(t)\}_{t=1}^{t=\infty}$ satisfying the following condition: for any t, $t = 1, 2, \dots$, there does not exist an admissible trajectory $\{X(t)\}_{t=1}^{t=\infty}$ such that $x_i(t) > \overline{x}_i(t)$ for all $i \in \{i\}$, for which $\overline{x}_i(t) > 0$. The trajectory $\{\overline{X}(t)\}_{t=1}^{t=\infty}$ will be called ∞-optimal.

II, 2. Denote by $\mathfrak{X}(\tau)$ the set of points $X \in R_{X_0,\tau}$ such that there exists an ∞-optimal trajectory $\{\overline{X}(t)\}_{t=1}^{t=\infty}$, admissible for x_0, for which $\overline{X}(\tau) = X$.

Theorem 3. Let an arbitrary Gale-type model $\mathfrak{M}(Z_t, f(t))$ and an initial state X_0 be given. Then: 1) there exists an ∞-optimal-

trajectory, i.e. the set $X(t)$ is nonempty for every t; 2) whatever the cost vector C, $X_T(t) \to X(t)$ as $T \to \infty$, for every t, $t = 1, 2, \dots$.

III, 1. A closed model $\mathfrak{M}(Z)$, in which the cone Z does not change over time, is called Gale' s model[6].

III, 2. In what follows, the notion of an equilibrium state of the model $\mathfrak{M}(Z)$ will be essential. An equilibrium state of the model $\mathfrak{M}(Z)$ is characterized by a process $(X, Y) \in Z$, a price vector $P = (p_1, \dots, p_n)$, $P \geq 0$, $\sum_i p_i > 0$, and a positive number α, which are in the following relations:

a) $\alpha\overline{X} \leq \overline{Y}$;

b) $(\overline{Y}, P) - (\overline{X}, \alpha P) = 0$;

c) $(Y, P) - (X, \alpha P) \leq 0$;

d) $(\overline{Y}, P) > 0$.

The number $\alpha = (\overline{Y}, P)/(\overline{X}P)$ is called the growth rate of the model $\mathfrak{M}$ in the corresponding equilibrium state. The equilibrium states, of which there are finitely many, are numbered by the index ν. $\alpha^{(\nu)}, P^{(\nu)}, (X^{(\nu)}, Y^{(\nu)})$ denote, respectively, the growth rate, the price vector, and the process determining the equilibrium state ν.

III, 3. Denote by $F_P^{(\nu)}$ the set of processes $(X, Y) \in Z$ satisfying the equation $(X, \alpha^{(\nu)} P^{(\nu)}) = (Y, P^{(\nu)})$, where $P^{(\nu)}$ is some price vector of the equilibrium state ν. The Neumann face $F^{(\nu)}$ of the cone Z corresponding to the equilibrium state ν is defined as the intersection of all $F_P^{(\nu)}$, i.e.

$$F^{(\nu)} = \bigcap_{P^{(\nu)} \in \{P^{(\nu)}\}} F_P^{(\nu)}.$$

Here $\{P^{(\nu)}\}$ is the set of all price vectors determining the equilibrium state ν.

III, 4. By $\rho(E, F)$ is denoted the Euclidean distance between the sets E and F. In addition, the following convention is introduced: if under the sign $\rho(\ldots)$ there stands a process (X, Y), then (X, Y) should be understood as a point, and not a ray, the normalization being determined by the relation $\sum_i x_i = 1$.

III, 5. The mean growth rate of product i in the sequence $\{X(t)\}_{t=1}^{t=\infty}$ is defined to be the largest positive number α_i for which there exist numbers $c_{\min}$ and $c_{\max}$ and a subsequence $\{t_k\}$ such that

$$c_{\min}\alpha_i^t \leq x_i(t) \leq c_{\max}\alpha_i^t$$

for all $t \in \{t_k\}$. We shall say that the mean growth rate of product i in the sequence $\{X(t)\}_{t=1}^{t=\infty}$ is less than α_i if there is no number $c_{\min}$ such that $c_{\min}\alpha_i^t \leq x_i(t)$ for all t.

Theorem 4. Let an arbitrary model $\mathfrak{M}(Z)$ and initial state X_0 be given. For any $\varepsilon > 0$ and any trajectory $\{X(t)\}$ admissible under X_0 such that, on some subsequence $\{t_k\}$,

$$\rho\big((X(t), X(t+1)), F^{(\nu)}\big) \geq \varepsilon,$$

the mean growth rate of each product $i \in \{i\}$ is less than $\alpha^{(\nu)}$.

III, 6. By $p_*^{(\nu)}$ is denoted an arbitrary price vector from the set $\{p^{(\nu)}\}$ that is not an extreme point of $\{p^{(\nu)}\}$.

Theorem 5. Let arbitrary $\mathfrak{M}(Z)$ and X_0 be given. Suppose, moreover, that for some trajectory $\{X(t)\}$ admissible under X_0 the mean growth rate of all products is $\leq \alpha^{(\nu)} - \delta$, while the mean growth rate of some product i_0 is equal to $\alpha^{(\nu)} - \delta$, and $p_{*,i_0}^{(\nu)} > 0$ for at least one $p_*^{(\nu)} \in \{p_*^{(\nu)}\}$. Then there exist a subsequence $\{t_l\}$ and a number $\varepsilon > 0$ such that

$$\rho\big((X(t), X(t+1)), F^{(\nu)}\big) \geq \varepsilon$$

on $\{t_l\}$.

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Note: Figure translations are in progress. See original paper for figures.