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Abstract

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A Method for Calculating Flows with Strong Viscous Interaction

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1. In calculating the flow of a supersonic stream of a viscous gas past bodies, in a number of cases it is impossible to confine oneself to the equations of the boundary layer or of an inviscid gas. The method proposed in the present paper makes it possible to obtain an approximate solution of the system of Navier–Stokes equations in such a way that the complete system has to be solved only in small regions in which the dissipative terms are large.

2. For the flows of interest to us the Reynolds number is a quantity of order 10^4—10^6; therefore the Navier–Stokes equations are equations with a small parameter ε multiplying the highest derivatives ($\varepsilon = 1/\mathrm{Re}$). Further, the system of equations of an ideal gas, obtained by discarding the highest derivatives, requires for its solution a smaller number of boundary conditions than the complete system, both on the body being flowed around and on the outer boundary. Therefore the model of aerodynamic problems is the following boundary-value problem A_ε for an ordinary differential equation with a small parameter ε multiplying the highest derivatives.

On the interval $[0, 1]$ it is required to solve the equation

$$L_\varepsilon u = (L_0 + \varepsilon L_1)u \equiv \sum_{j=0}^{k} a_j(x)\frac{d^j u}{dx^j} + \sum_{r=1}^{l} \varepsilon^r a_{k+r}(x)\frac{d^{k+r}u}{dx^{k+r}} = h(x); \qquad (1)$$

$$k = k_1 + k_2; \qquad l = l_1 + l_2,$$

with boundary conditions

$$\left.\frac{d^i u}{dx^i}\right|_{x=0} = 0; \qquad i = 0, 1, \dots, k_1 + l_1 - 1; \qquad (2)$$

$$\left.\frac{d^j u}{dx^j}\right|_{x=1} = 0; \qquad j = 0, 1, \dots, k_2 + l_2 - 1. \qquad (3)$$

We shall call the solution of the equation $L_0 u = h$, with the smaller number of conditions of the form (2)–(3) at the points $x = 0$, $x = 1$, the **degenerate problem** A_0: k_1 conditions at the left end and k_2 at the right end of the interval.

The approximate solution of the problem A_ε is a function w, which is determined as follows: $w = w_1$ for $x \in [0, \bar{x}_1]$, where w_1 is the solution of the equation $L_\varepsilon w_1 = h$ on the interval $[0, \bar{x}_1]$ with conditions (2) at $x = 0$ and conditions $d^i w_1/dx^i = a_i$, $i = 0, 1, \dots, k_2 + l_2 - 1$, at $x = \bar{x}_1$.

$w = w_2$ for $x \in [\bar{x}_1, \bar{x}_2]$, where w_2 is the solution of the equation $L_0 w_2 = h$ on the interval $[\bar{x}_1, \bar{x}_2]$ with conditions $d^j w_2/dx^j\big|_{x=x_1} = b_j$, $j = 0, 1, \dots, k_1 - 1$; $d^r w_r/dx^r\big|_{x=\bar{x}2} = c_r$, $r = 0, 1, \dots, k_2 - 1$.

$w = w_3$ for $x \in [\bar{x}_2, 1]$, where w_3 is the solution of the equation $L_\varepsilon w_3 = h$ with conditions (3) at $x = 1$ and conditions $d^l w_3/dx^l\big|_{x=\bar{x}2} = d_l$, $l = 0, 1, \dots, \dots, k_1 + l_1 - 1$.

The constants a_i, b_j, c_r, d_l (their number is $2(k_2 + k_1) + l_2 + l_1$) are found from the smoothness matching conditions (4)–(5) at the points $\bar{x}_1, \bar{x}_2$ ($\bar{x}_1 > x_1$, $\bar{x}_2 < x_2$, $x_1 < x_2$); the position of the points x_1 and x_2 is determined by fulfilling the requi-

condition 2) of Theorem 1),

$$d^j w_1/dx^j\big|_{x=\bar{x}1} = d^j w_2/dx^j\big|_{x=\bar{x}1}, \qquad j = 0, 1, \dots, k_1 + k_2 + l_2 - 1; \tag{4}$$

$$d^i w_2/dx^i\big|_{x=\bar{x}2} = d^i w_3/dx^i\big|_{x=\bar{x}2}, \qquad i = 0, 1, \dots, k_1 + k_2 + l_1 - 1. \tag{5}$$

Denote by $\lambda_i(x)$ the roots of the equation $\sum_{r=0}^{l} a_{k+r}(x)\lambda^r = 0$, considered with respect to the unknown λ.

Theorem 1. Suppose: 1) the problem A_0 is solvable; 2) inside the interval $[0, 1]$ there exist points $x = x_1$, $x = x_2$, $x_1 < x_2$, such that for $x \in [x_1, x_2]$, among the $\lambda_i(x)$ there are l_1 functions $-\lambda_1, \dots, -\lambda_{l_1}$ with negative real parts, and $l_2 = l - l_1$ functions $\lambda_{l_1+1}, \dots, \lambda_l$ with positive real parts, which coincides with the number of conditions additional to the conditions of the problem A_0, respectively at the left and right ends of the interval; 3) the problem A_ε, $L_\varepsilon u = g$, under conditions (2)–(3) is solvable, its solution is unique, and for any ε

$$\|u\| < k\|g\|,$$

where k does not depend on ε.

Then there exist $\bar{x}_1 > x_1$, $\bar{x}_2 < x_2$, such that the approximate solution w exists, is unique, and differs from the exact one by no more than a quantity of order $o(\varepsilon)$.

The approximate solution w, constructed by the method described, coincides with the first approximation of [1] in the case when the problem A_ε degenerates regularly into A_0 in the sense of [1]. However, the method described makes it possible to obtain an approximate solution also in those cases when there is no regular degeneration in the sense of [1], for example, in a problem with an initial jump (initial conditions of the form $1/\varepsilon^k$), and also in all those cases when the coefficients of the equation are such that its solution behaves in a complicated way near the ends of the interval $[0, 1]$ and only inside the interval $[\bar{x}_1, \bar{x}_2]$, $\bar{x}_1 > x_1$, $\bar{x}_2 < x_2$, tends, as $\bar{x}_1/\varepsilon \to \infty$, $(-\bar{x}_2)/\varepsilon \to \infty$, to some smooth function, and this smooth function may differ from the solution of the degenerate problem A_0 by a finite amount. Such behavior of the solution occurs if, for $x \in [0, x_1)$ and $x \in (x_2, 1]$, the signs of the real parts of the functions $\lambda_i(x)$ do not satisfy requirement 2) of Theorem 1. This is a model of such gas-dynamic problems as, for example, the calculation of the flow near the nose of a sharpened plate; the calculation of the flow in the region of separation of the boundary layer arising as a result of the interaction of a shock wave with the boundary layer; the calculation of flow past a step. In the cases listed, the flow parameters are functions that behave in a complicated manner in some region near the body (separation), while in the outer region they differ by a finite amount from the solution of the degenerate problem, i.e., from the parameters of simply inviscid flow (strong viscous interaction).

§ 3. The following model is a partial differential equation of n-th order with a small parameter ε at the highest derivatives.

Let the problem A_ε consist in solving an equation of $(k + l)$-th order

$$L_\varepsilon u = L_0 u + \sum_{s=1}^{l} \varepsilon^s L_{k+s} u = 0 \tag{6}$$

in a domain Q (by L_s is denoted a linear differential operator of s-th order). On different parts Γ_i of the boundary Γ of the domain Q there are prescribed, generally speaking, different nonhomogeneous boundary conditions. Suppose that the domain Q can be divided into m parts $Q_1, Q_2, \ldots, Q_m$ in such a way that in certain Q_i $(1 \leq i \leq m)$ the problem A_ε is expressed in terms of simpler problems $A_{\varepsilon i}$, in the sense that the exact solution u of the problem A_ε in the domains Q_i satisfies, with accuracy up to quantities of order $o(\varepsilon)$, certain simpler equations

$L_{\varepsilon i} u = 0$ with the corresponding boundary conditions for them. Suppose that u in the remaining subdomains Q_j does not satisfy any simplified equation and can be found only as the solution of equation (6) with conditions analogous to the conditions on Γ in problem A_ε.

The approximate solution is defined as follows: $w = w_i$ in the regions Q_i, where each w_i is a solution of the corresponding equation $L_{\varepsilon i} w_i = 0$. If in the given Q_j equation (6) cannot be simplified, then w_j is a solution of equation (6). On the

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

boundary of each of the subdomains, conditions are prescribed corresponding to the equation that is being solved in it. On those parts of the boundary that coincide with the boundary of the region Q, the conditions coincide with the conditions on the corresponding parts Γ_i in problem A_ε. On those parts of the boundaries Γ_j of the subdomains Q_i that lie inside Q, the right-hand sides of the corresponding inhomogeneous conditions are unknown (obviously, the number and type of conditions on each Γ_j depend on the form of this line). We set the right-hand sides equal to certain unknown functions φ_j and prescribe on each Γ_j as many smooth matching conditions (i.e., continuity of the function w and of the necessary number of its normal derivatives to Γ_j) as are needed for the unique determination of all unknown φ_j prescribed on Γ_j.

Fig. 1

Fig. 2

Theorem 2. *If: 1) the problems A_ε, $A_{\varepsilon i}$ are uniformly solvable with respect to ε; 2) the system obtained for determining φ_j is uniquely solvable, then the approximate solution w exists, is unique, and differs from the exact one by a quantity of order $O(\varepsilon)$.*

For several examples of partial differential equations, an approximate solution w has been constructed according to this algorithm, and it has been proved that it is unique and differs from the exact solution by no more than a quantity $o(\varepsilon)$.

Example. In the circle Q, $\rho \le a$, the equation

$$L_\varepsilon u \equiv \varepsilon \Delta u + \frac{\partial u}{\partial x} = h(x, y)$$

is given with the boundary condition $u|_\Gamma = 0$. Let us divide the region Q by the line Γ_3 into two parts Q_1 and Q_2 so that everywhere in Q_2, $|\varepsilon \Delta u| = O(\varepsilon)$, if u is the exact solution of the problem (the possibility of such a division is proved in (1)).

The approximate solution in Q_1 (Fig. 1), $w = w_1$, is the solution of the equation $L_\varepsilon w_1 = h$ with the conditions $w_1|_{\Gamma_1} = 0$; $w_1|_{\Gamma_3} = \varphi_1$; in Q_2, $w = w_2$ is the solution of the equation $\partial w_2/\partial x = h$ with the condition $w_2|_{\Gamma_2} = 0$. The unknown function φ_1 is found from the condition of continuity of w on Γ_3, i.e. $w_1|_{\Gamma_3} = w_2|_{\Gamma_3}$. The constructed approximate solution w is such that $|u - w| = O(\varepsilon)$ everywhere in the circle $\rho \le a$.

4. The algorithm for constructing an approximate (with accuracy up to quantities $o(\varepsilon)$) solution of the Navier–Stokes equations is analogous to the algorithm for a single partial differential equation.

For example, suppose it is required to find the flow parameters for the supersonic flow of a viscous gas past an angle less than π. The approximate solution of the Navier–Stokes equations is sought in the region $Q = Q_1 + Q_2 + Q_3 + Q_4$ (Fig. 2), bounded on the left by a vertical straight line, on which all parameters of the incident flow are known, and above by the characteristic Γ_1 of the system of inviscid-gas equations.

The approximate solution in Q_4, $w = w_4$, is a solution of the system of inviscid-gas equations; in Q_2 and Q_3, $w = w_2$, $w = w_3$ are solutions of the system of equations

boundary layer; in Q_1, $w = w_1$ is the solution of the complete Navier–Stokes system. The nature of the equations solved in the regions Q_4 and Q_2 is such that no conditions are required on the boundaries Γ_1 and Γ_2 for finding w_4 and w_2.

The Navier–Stokes equations, the boundary-layer equations, and the inviscid-gas equations are solved jointly throughout the entire region Q, with the necessary number of smooth matching conditions prescribed on the boundaries of the regions Q_1, Q_2, Q_3, and Q_4 lying inside Q. This ensures that the mutual influence of the flows near the body and in the outer region is taken into account. The boundaries of the regions Q_1, Q_2, Q_3, Q_4 are chosen so that the terms of the Navier–Stokes equations discarded in Q_2, Q_3, Q_4 are quantities of order $O(\varepsilon)$.

Thus, the complete Navier–Stokes system need be solved only in the immediate neighborhood of the corner point. Similarly, in computing the flow past a sharpened plate, the complete Navier–Stokes system need be solved only in the neighborhood of the "nose," etc. A difference method is proposed for the numerical solution of the resulting boundary-value problems.

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REFERENCES

1. M. I. Vishik, L. A. Lyusternik, UMN, **12**, no. 5 (77), 3 (1957).

Note: Figure translations are in progress. See original paper for figures.