ON THE STRUCTURE OF SOLUTIONS OF THE CHAIN OF EQUATIONS OF THE KINETIC THEORY OF GASES

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Abstract

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ON THE STRUCTURE OF SOLUTIONS OF THE CHAIN OF EQUATIONS OF THE KINETIC THEORY OF GASES

In recent years the kinetic theory of gases, in particular the Boltzmann kinetic equation, has been applied very successfully to the solution of many modern problems of aerodynamics. In deriving this equation a number of simplifying assumptions were made that were not properly substantiated (^{1}). A major contribution to the further development of the kinetic theory of gases and to the substantiation of the kinetic equation was made in the works of N. N. Bogolyubov (^{2}), M. Born and H. Green (^{3}), and J. Kirkwood (^{4}), in which the Boltzmann equation was obtained from a chain of kinetic equations. In recent years new works devoted to the substantiation of kinetic theory have appeared —those of H. Hollinger and Curtiss (^{5}), Freiman (^{6}), and others—in which, in deriving the kinetic equation, certain assumptions were also made concerning the structure of the correlation functions, their dependence on time, and so forth. It seems advisable to continue such investigations, in particular in the directions indicated by Uhlenbeck (^{7}).

In the present work the structure of solutions of the chain of kinetic equations is considered, and it is shown that in the equation for the s-particle distribution function, in addition to the usual type of interaction, there appears a special integral type of interaction, not characteristic of classical mechanics, since in this case the molecules behave as nonlocalized particles. This type of interaction plays an important role in the dynamics of irreversible processes. In particular, it is shown in the work that the usual method of successive approximations leads to a fully reversible solution for the chain of equations. In this case the integral interaction terms vanish identically. To construct irreversible solutions it is necessary, in the zeroth approximation, to take into account the role of the integral interaction terms. It is shown in the work that the correlation distribution functions, in accordance with the two indicated types of interaction, must depend simultaneously on the fast and slow processes occurring in the system. Accordingly, for studying solutions of the chain of kinetic equations, the work applies a modified method of a small parameter, analogous to the method used by the author in (^{8}), which made it possible to reduce the chain of kinetic equations to a system of partial differential equations.

Let us consider an isolated system consisting of N identical monatomic molecules located in a volume V. The distribution function of such a system satisfies the Liouville equation

$$\partial F_N/\partial t = \mathcal{H}_N F_N. \tag{1}$$

The operator $\mathcal{H}_N$ is defined by the expression:

$$\mathcal{H}_N = -\sum_{i=1}^{N} \frac{\mathbf{P}_i}{m}\frac{\partial}{\partial \mathbf{r}_i} + \sum_{i<j}\sum U_{ij}, \tag{2}$$

where

$$U_{ij} = \frac{\partial \Phi\left(|\mathbf{r}_i - \mathbf{r}_j|\right)}{\partial \mathbf{r}_i}\frac{\partial}{\partial \mathbf{P}_i} + \frac{\partial \Phi\left(|\mathbf{r}_i - \mathbf{r}_j|\right)}{\partial \mathbf{r}_j}\frac{\partial}{\partial \mathbf{P}_j}.$$

The distribution function for the entire system is usually normalized to unity.

Equation (1) is equivalent to the corresponding system of Hamilton equations. However, statistics is introduced here into the initial data. Therefore equation (1) will determine, in a completely reversible manner, the process of evolution in time of the initial probability distributions. The distribution function for groups consisting of s molecules is defined by the expression:

$$F_s(t, x_1, \dots, x_s) = V^s \int \cdots \int F_N \, dx_{s+1} \dots dx_N. \tag{3}$$

For these s-particle distribution functions, a chain of equations can be obtained by integrating the original Liouville equation with respect to the variables $dx_{s+1}, dx_{s+2}, \dots, dx_N$. Integrating equation (1), to which the operator (2) has been applied, we obtain

$$\frac{\partial F_s}{\partial t} = \mathcal{H}_s F_s + \frac{N-s}{V}\sum_{k=1}^{s}\iint U_{ks+1} F_{s+1}\, dx_{s+1}. \tag{4}$$

In this equation, between the s molecules of the group and the remaining $N-s$ molecules of the system, a special type of interaction has appeared, expressed by the integral terms of equation (4). For $s=1$ they can be written in the form: $\operatorname{div}_v F_1(t,r,V) D_1(t,r,V)$, where

$$D_1(t,r,V) = \frac{N-1}{m}\iint \operatorname{grad}_{r_1} \Phi(|r_1 - r_2|) F_2(t, x_1, x_2)\, dx_2 \Big/ \iint F_2(t, x_1, x_2)\, dx_2.$$

As is seen, D_1 is the interaction force between the first and second molecules, averaged over the phase space of the second molecule. Such a type of interaction between particles does not exist in classical mechanics. The molecules behave as nonlocalized particles, and the correlation between them leads to a dependence of the interaction force on the velocity. This type of interaction plays the principal role in the dynamics of irreversible processes. Let us note that a similar type of interaction between particles, under the additional assumption of the absence of correlation between them, was first introduced into statistics in the works of A. Vlasov (^{9}). Usually in equations (4) a purely formal transition is made to systems with $N \to \infty$ and $V \to \infty$, but with finite density.* We shall not make this transition and shall assume that the parameters of the system N and V may take arbitrarily large, but finite, values. In equations (4), before the sum there stands the small parameter $\frac{N-s}{L} r_0^3$, and therefore for their solution one may apply the usual method of successive approximations:

$$\frac{\partial F_s^{(0)}}{\partial t} = \mathcal{H}_{sF} s^{0},$$

$$\frac{\partial F_s^{(1)}}{\partial t} = \mathcal{H}_{sF} s^{(1)} + \frac{N-s}{V} \sum_{k=1}^{s} \iint U_{ks+1} F_{s+1}^{(0)} \, dx_{s+1}, \tag{5}$$

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In the zeroth approximation we shall have the solution

$$F_s^{(0)}(t) = e^{t\mathcal{H}_s} F_s^{(0)}(0) = F_s^0(0, X_1(t), \dots, X_s(t)), \tag{6}$$

where the functions $X_k(t) = e^{t\mathcal{H}_s} x_k$ are the solution of the corresponding Hamilton equations. Consequently, for arbitrary initial data in the zeroth approximation we obtain a completely reversible solution. In the first approximation the distribution function satisfies the inhomogeneous equation (5).

We write the inhomogeneous term of this equation in the form:

$$\Psi_s^{(1)} = -\frac{N-s}{V} \left\{ \iint \left(\mathcal{H}_{s+1} F_{s+1}^{(0)} - H_{sF_{s+1}}^{(0)} + V_{s+1} \frac{\partial F_{s+1}^{(0)}}{\partial r_{s+1}} \right) dx_{s+1} \right\}.$$

After transformations we find

$$\Psi_s^{(1)} = (N-s)\{\partial F_s^{(0)}/\partial t - \mathcal{H}_{sF} s^{(0)}\} = 0.$$

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* In this case, (4) is called the Bogolyubov chain of equations.

Consequently, in the first approximation the distribution function is also determined by the homogeneous Liouville equation and coincides with the zeroth approximation. All subsequent approximations will likewise coincide with one another and will lead to the exact, fully reversible solution for the chain of equations (4).

As is evident, in the usual method of successive approximations the zeroth approximation coincided with this exact solution, and the subsequent approximations did not lead to new results. Therefore, here it is necessary to apply more effective methods of solution. For further investigation of the system (4), let us introduce another normalization of the distribution functions and adopt new notation:

$$f_1(t_1, x_1) = \frac{N}{V} F_1(t, x_1); \qquad f_2(t, x_1, x_2) = \frac{N(N-1)}{V^2} F_2(t, x_1, x_2),$$

$$f_s(t, x_1, \dots, x_s) = \frac{N(N-1)\dots(N-s+1)}{V^s} F_s(t, x_1, \dots, x_s).$$

In this case the chain of kinetic equations can be written in the form

$$\begin{aligned} \frac{\partial f_1}{\partial t} &= -V_1 \frac{\partial f_1}{\partial r_1} + \varepsilon \iint W_2 f_2 \, dx_2, \\ \frac{\partial f_2}{\partial t} &= \mathcal{H}_2 f_2 + \varepsilon \iint W_3 f_3 \, dx_3, \\ \frac{\partial f_3}{\partial t} &= \mathcal{H}_3 f_3 + \dots, \\ &\dots\dots\dots\dots\dots\dots \end{aligned} \tag{7}$$

Here the factor $(N-s)/V$ has disappeared before the integral, but the small parameter of order r_0/λ, naturally, has remained.

Let us now consider how the two types of interaction between molecules indicated above can affect the structure of the correlation distribution functions. As is seen from (7), the interaction within a group of molecules is expressed by the differential terms of the equations (drift terms). This interaction can lead to a very substantial change in the distribution function over small time intervals of the order of the collision time $\tau_0 \approx r_0/c$. At the same time, the interaction of a given group of molecules with the remaining molecules of the system is expressed by the integral terms of the equation with the small parameter ε. This integral interaction can lead to a significantly slower change in the distribution functions over time intervals of order τ_0. The relation between the two types of interaction will be different for different distribution functions. For the one-particle distribution function $f_1(t, r, v)$, equation (7) contains only the integral interaction term. Therefore this distribution function will be a slowly varying function of time. For all the remaining correlation functions with $s \geq 2$, equations (7) will contain both integral and differential interaction terms. Therefore these correlation functions will depend on time in a complicated way. On the one hand, they may change rapidly over intervals of the order of the collision time; on the other hand, they will depend on functions that change slowly over such a time interval. These propositions are basic for the subsequent analysis. In particular, for these reasons it seems necessary to apply, for solving the chain of equations (7), the modified method of the small parameter (8).

Let us represent the correlation functions f_s for $s \geq 2$ in the form of power expansions:

$$f_s = f_s^{(0)} + \varepsilon f_s^{(1)} + \varepsilon^2 f_s^{(2)} + \varepsilon^3 f_s^{(3)} + \cdots . \tag{8}$$

Moreover, in accordance with what was set forth above, we shall take

$$f_s(t) = f_s(t_1, x_1, \dots, x_s, \omega_1, \dots, \omega_s).$$

With respect to the set of functions ω_k, for the time being we assume only that

$$\partial \omega_k / \partial t = \varepsilon A_k + \varepsilon^2 B_k + \varepsilon^3 C_k + \cdots , \tag{10}$$

where $A_k, B_k, \dots$ are likewise sets of as yet unknown functions. In this case the distribution function will depend explicitly on the "fast" time and functionally on the slow time through the dependence of the functions ω_k on t.

Substituting (8), (9), and (10) into (7) and collecting terms with equal powers of ε, we obtain the recurrent system of equations

$$\left(\frac{\partial f_s^{(0)}}{\partial t}\right)_\omega = \mathcal{H}_s f_s^{(0)},$$

$$\left(\frac{\partial f_s^{(1)}}{\partial t}\right)_\omega = \mathcal{H}_s f_s^{(1)} - A_k \frac{\partial f_s^{(0)}}{\partial \omega_k} + \iint W_{s+1} f_{s+1}^{(0)} \, dx_{s+1}, \tag{11}$$

$$\left(\frac{\partial f_s^{(2)}}{\partial t}\right)_\omega = \mathcal{H}_s f_s^{(2)} - A_k \frac{\partial f_s^{(1)}}{\partial \omega_k} - B_k \frac{\partial f_s^{(0)}}{\partial \omega_k} + \iint W_{s+1} f_{s+1}^{(1)} \, dx_{s+1},$$

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This system can serve to determine successive approximations f_s after the set of unknown functions $\omega_k, A_k, B_k, \dots$ has been determined. In accordance with the method applied here, there is a certain arbitrariness in the definition of ω_k, since it is necessary to satisfy only the general condition (10). Therefore these functions should be chosen in such a way as to take fuller account of the dynamical properties of the system and, as far as possible, to simplify its study. In particular, it is expedient to express the functions ω_k in terms of functions characteristic for the given problem. Up to now only the first equation (7), which determines the one-particle distribution function, has not been used. This is the only function of the given problem which, like ω_k, changes little on time intervals of the order of the collision time. Therefore we define the set of unknown functions as follows:

$$\frac{\partial \omega_k}{\partial t} = \left(\frac{\partial f_1(t,\xi,V_k)}{\partial t} + V_k \frac{\partial f_1}{\partial \xi}\right)_{\xi=r_k+V_k t} \varepsilon = \varepsilon \iint W_2 f_2 \, dx_2 \Big|_{\xi=r_k+V_k t} . \tag{12}$$

In this case we shall have

$$\omega_k(t, r_k, V_k) = f_1(t, r_k + V_{kt}, V_k).$$

Moreover, we obtain the following expressions for the unknown functions

$$A_k(r_k, V_k, t) = \iint W_2(\xi, V_k, x_2)\, f_2^{(0)}(t, \xi, V_k, x_2)\, dx_2 \Big|_{\xi=r_k+V_{kt}}$$

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Now, in the recurrent system of equations (11), all auxiliary functions have been determined, and, taking into account the established structure of the solutions, it can serve to determine successive approximations f_s. These questions will be considered in a separate paper.

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Note: Figure translations are in progress. See original paper for figures.