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Abstract

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MATHEMATICS

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ON THE QUESTION OF THE FORM OF EXTREMAL NONNEGATIVE POLYNOMIALS WHOSE COEFFICIENTS ARE SUBJECT TO SEVERAL LINEAR RELATIONS

(Presented by Academician S. N. Bernstein on December 4, 1964)

1. In solving extremal problems in the theory of monotone polynomials that deviate least from zero on a certain interval, it is important to know the form of the polynomials among which the extremal ones are contained.

Among the fundamental results obtained in this direction one should include, first of all, the well-known theorems of S. N. Bernstein ⁽¹⁾ on nonnegative trigonometric polynomials, and also, as a development of S. N. Bernstein's ideas, the theorems of B. A. Rymarenko ⁽²⁻⁴⁾ on monotone and multiply monotone algebraic polynomials (and certain monotone functions of other classes). However, in these theorems only the case is considered in which the number of relations between the coefficients of the polynomials does not exceed two.

A number of extremal properties of algebraic and trigonometric nonnegative polynomials (and of certain other functions), which make it possible to refine the form of extremal polynomials also in the case when the coefficients of the polynomials satisfy any number of linear relations, were established in our communications ⁽⁵⁻⁷⁾. The present communication is devoted to the same circle of questions.

2. Consider the problem:

Among all polynomials of degree not exceeding n with real coefficients

$$y_n(x) = \sum_{k=0}^n p_k x^k, \quad (1)$$

which are nonnegative on a prescribed interval E (finite or infinite) of the real axis and satisfy s relations linear with respect to the coefficients p_k ,

$$\omega_j\{y_n(x)\} \equiv \sum_{k=0}^n p_k a_{kj} = A_j, \quad j = 1, \dots, s, \quad (2)$$

where a_{kj}, A_j are given real numbers

$$\left(\sum_{j=1}^s |A_j| > 0 \right),$$

to find that polynomial which realizes the minimum of the integral

$$\int_a^b p(x) y_n(x) dx. \quad (3)$$

Here $[a, b] \subset E$, and $p(x)$ is a given nonnegative function integrable on $[a, b]$.*

* Such a problem leads, in particular, to the problem of finding the minimal oscillation on the segment $[a, b]$ of the function

$$f_n(x) = \int_a^x (x-t)^h p(t) y_n(t) dt$$

(monotone or multiply monotone) satisfying s linear relations. Here $h \geq 0$ is a given integer; the remaining notation is as before.

Represent the polynomial $y_n(x)$ in the form

$$y_n(x) = \varphi(x) u^2(x) q(x), \quad (4)$$

where $\varphi(x)$ is a polynomial of degree not exceeding two, whose roots (moreover simple ones) can be only the endpoints of the interval E ; $u(x)$ is a polynomial all of whose roots are real and belong to the interval E ; $q(x)$ is a polynomial having no real roots in E . Denote the degrees of the polynomials $u(x)$ and $q(x)$ by m and r , respectively.

Then, if $y_n(x)$ is an extremal polynomial, then, as was proved earlier (5), $r \leq s - 1$, and for each integer τ , $0 \leq \tau \leq m + r - s$, there exists a polynomial $t(x; \tau)$ of degree not exceeding s such that the equalities

$$\int_a^b p(x) \varphi(x) t(x; \tau) u(x) x^\tau dx = 0;$$

$$\omega_j\{\varphi(x) u(x) t(x; \tau) x^\tau\} = 0, \quad j = 1, \dots, s.$$

3. We shall now restrict ourselves to problems with such constraints (2) for which there exists a polynomial $t(x)$, nonnegative on $[a, b]$, of degree not exceeding s , satisfying the relations

$$\omega_j\{\varphi(x)t(x)x^k\} = 0^*, \quad j = 1, \dots, s; \quad k = 0, \dots, 2m + r - s.$$

Then for all values of τ we may put

$$t(x; \tau) = t(x),$$

and the coefficients of the polynomial $u(x)$ will satisfy the system of equations

$$\int_a^b \omega(x)u(x)x^\tau dx = 0, \quad \tau = 0, \dots, m + r - s, \quad (5)$$

where $\omega(x) = p(x)\varphi(x)t(x)$.

Construct a system of polynomials $\{Q_k(x)\}$, $k = 0, 1, \dots$, orthogonal on the segment $[a, b]$ with respect to the weight $\omega(x)$, and expand the polynomial $u(x)$ in these polynomials:

$$u(x) = \sum_{k=0}^m \alpha_k Q_k(x).$$

From the system of equations (5) we obtain that

$$\alpha_0 = \alpha_1 = \dots = \alpha_{m+r-s} = 0,$$

and the solution of the problem reduces to finding from the relations (2) the $s - r$ remaining coefficients of $u(x)$ and the r coefficients of $q(x)$.

Of special interest is, evidently, the case when $r = s - 1$ and, consequently, $u(x) = a_m Q_m(x)$.

We note that all roots of $u(x)$ in this case are simple and belong to the interval (a, b) . For any value of r , however, such a conclusion obviously cannot be drawn; but even then the number of multiple roots of $u(x)$ and roots lying outside (a, b) does not exceed a certain number. Namely, the following is true.

Lemma 1. Represent the polynomial $u(x)$ in the form

$$u(x) = (x - \eta_1)^{2k_1} \dots (x - \eta_m)^{2k_m} \prod_{i=1}^M (x - \eta_i),$$

* Such problems are, for example, problems in which an arbitrary number of consecutive leading coefficients of $y_n(x)$, the values of $y_n(x)$, and its consecutive first derivatives at some prescribed point are given.

where $\eta_1, \eta_2, \dots, \eta_\mu$ are distinct roots of odd multiplicity (including simple ones). Further, suppose that among the roots η_i , $i = 1, \dots, \mu$, there are μ_1 roots lying outside (a, b) . Then, if the system of equations (5) is satisfied, then

$$2 \sum_{i=1}^m k_i + \mu_1 \leq \delta, \quad \delta = s - r - 1.$$

4. Let us take L arbitrary pairwise distinct numbers x_1, x_2, \dots, x_L belonging to the interval (a, b) , and form the polynomial

$$V_L(x) = \prod_{i=1}^L (x - x_i).$$

Consider the function

$$V_L(x; t) = \frac{V_L(x) - V_L(t)}{V_L'(t)(x - t)},$$

which, for each fixed value of t , is also a polynomial. Note that $V_L(t; t) = 1$, and let us formulate a series of lemmas established for these functions.

Lemma 2. Among the numbers

$$\int_a^b \omega(x) \Phi(x) V_L(x; x_i) dx, \quad i = 1, \dots, L, \quad (6)$$

where $\Phi(x)$ is any polynomial nonnegative on $[a; b]$, there exist numbers not equal to zero.

Lemma 3. Whatever polynomial $\Phi(x)$, nonnegative on $[a; b]$, may be, all the numbers (6) will either be equal to M/L , where

$$M = \int_a^b \omega(x) \Phi(x) dx,$$

or among them there will be at least one number that is less than M/L .

5. Now suppose that $x_i = \eta_i$, $i = 1, \dots, l \leq L$, where η_i are some simple roots of the polynomial $u(x)$ lying inside (a, b) .

In what follows we shall adhere to the notation

$$V_l(x) = \prod_{i=1}^l (x - \eta_i); \quad W_{m-l}(x) = \frac{u(x)}{V_l(x)};$$

$$V_L(x) = V_l(x) \prod_{i=l+1}^L (x - x_i)$$

(no restrictions are imposed for the time being on the numbers x_i , $i = l + 1, \dots, L$),

$$I(x_i) = \int_a^b \omega(x) W_{m-l}^2(x) V_L(x; x_i) dx, \quad i = 1, \dots, L. \quad (7)$$

Then the following are valid.

Lemma 4. If $L \geq \delta$, $l \geq (L + \delta)/2$, where $\delta = s - r - 1$, and the system of equations (5) is satisfied, then, whatever the numbers x_i , $i = l + 1, \dots, L$, may be,

$$I(x_i) = 0, \quad i = l + 1, \dots, L.$$

Lemma 5. If $y_n(x)$ is an extremal polynomial, then there does not exist a polynomial $V_L(x)$, $L \geq \delta$, $l \geq (L + \delta)/2$, for which at least one number (7) would be negative.

Lemma 6. If $y_n(x)$ is an extremal polynomial, then, whatever the polynomial $V_L(x)$, $L \geq \delta$, $l \geq (L + \delta)/2$, and the number $l_1 \leq l$, among the numbers (7) there does not exist a set of l_1 numbers each of which would be

$$\geq M/l_1, \quad \text{where} \quad M = \int_a^b \omega(x) W_{m-l}^2(x) dx,$$

and at least one of them would be

$$> M/l_1.$$

The proof of the following assertion is based on these lemmas.

Theorem. For any $\varepsilon > 0$, to each extremal polynomial $y_n(x)$, starting from sufficiently large n , one can put in correspondence a comparison polynomial of the form

$$\tilde{y}_n(x) = \varphi(x) \tilde{u}^2(x) \tilde{q}(x),$$

such that

$$\int_a^b p(x)\tilde{y}_n(x) dx < \int_a^b p(x)y_n(x) dx + \varepsilon,$$

and the degree of the factor $\tilde{q}(x)$ will be equal to

$$\tilde{r} = r + 2\delta = s - 1 + \delta.$$

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