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Abstract

Full Text

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ON RELATIONS BETWEEN SOLUTIONS OF A SINGULAR CAUCHY PROBLEM CONCERNING A GENERALIZED DIFFERENTIAL-OPERATOR EULER-POISSON-DARBOUX EQUATION

(Presented by Academician I. N. Vekua on 1 VIII 1964)

1. In a number of recent works ([1–9]), for the equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{k}{t}\frac{\partial u}{\partial t} + cu = Xu \tag{1}$$

(k and c are constants), in which X is the Laplace operator with respect to the variables $x_1, \dots, x_m$ ([1–6]), or, more generally ([7,8]), a linear differential operator of second order of elliptic type in the same variables, the singular Cauchy problem

$$u\Big|_{t=0} = f(x), \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0; \qquad x = (x_1, \dots, x_m). \tag{2}$$

was studied.

Independently of the nature of the linear operator X in the variables x, in ([9]) formulas were established expressing the solution $u(t, x; k, c; f)$ of problem (1)–(2) in terms of $u(t, x; k', c'; f)$.

In the present work there are established (Sec. 2) the corresponding relations between solutions of the more general problem

$$\frac{\partial^n \tilde{u}}{\partial t^n} + \frac{k_1}{t}\frac{\partial^{n-1}\tilde{u}}{\partial t^{n-1}} + \dots + \frac{k_{n-1}}{t^{n-1}}\frac{\partial \tilde{u}}{\partial t} + c\tilde{u} = X\tilde{u}, \tag{3}$$

$$\tilde{u}\Big|_{t=0} = f(x); \qquad \frac{\partial^l \tilde{u}}{\partial t^l}\Big|_{t=0} = 0, \quad l = 1, \dots, n-1; \quad n \geqslant 2, \tag{4}$$

corresponding to two systems of values of the constants $k_1, \dots, k_{n-1}, c$. Also indicated (Sec. 4) is a formula expressing the solution of the Cauchy problem for equation (3) with $n = 1$ in terms of the solution of problem (3)–(4) with $n > 1$. In Sec. 5 some applications of the formulas obtained are given.

2.1. Equation (3) can be written in the form

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$$\left[t^{-n}\delta(\delta+n(b_1-1))\cdots(\delta+n(b_{n-1}-1))+c\right]u=Xu, \tag{3'}$$

where $\delta=tD,\ D=\partial/\partial t$, and the numbers

$$n(b_i-1)=\rho_i, \qquad i=1,\dots,n-1, \tag{4'}$$

are found from $k_1,\dots,k_{n-1}$ from the system

$$\sum_{i_1,\dots,i_s=1}^{n-1}\rho_{i_1}\cdots\rho_{i_s}+a_1^{(n+1-s)}\sum_{i_1,\dots,i_{s-1}=1}^{n-1}\rho_{i_1}\cdots\rho_{i_{s-1}}+\dots+a_{s-1}^{(n-1)}\sum_{i_1=1}^{n-1}\rho_{i_1}+a_s^{(n)}=k_s, \qquad s=1,\dots,n-1, \tag{5}$$

whose coefficients are determined by the representation

$$\delta^p=t^pD^p+a_1^{(p)}t^{p-1}D^{p-1}+\dots+a_{p-2}^{(p)}t^2D^2+tD, \qquad p=1,\dots,n. \tag{6}$$

The solution $\tilde{u}(t,x_1,\dots,x_m;k_1,\dots,k_{n-1},c;f)$ of problem (3)–(4) as a function of the parameters $b_1,\dots,b_{n-1},c$ will be denoted by $u(t,x_1,\dots,x_m;b_1,\dots,b_{n-1},c;f)$ (more briefly: $u(t,x;b_1,\dots,b_{n-1},c;f)$).

2.2. Suppose that for the values under consideration of the parameters $b_1,\dots,b_{n-1},c$ and of the given function $f(x)$ there exists a solution of problem $(3')$–(4). The following relations hold:

$$u(t,x;b_1+\beta_1,\dots,b_{n-1}+\beta_{n-1},c+\gamma;f)= \tag{7}$$

$$=A_n\int_0^1\cdots\int_0^1 K(t,\xi;\tilde{b},\tilde{\beta},\gamma)\,u\Big(t\sqrt[n]{\xi_1\cdots\xi_{n-1}},x;\ b_1,\dots,b_{n-1},c;f\Big)\,d\xi_1\cdots d\xi_{n-1},$$

$$K(t,\xi;b,\beta,\gamma)=$$

$$={}_0F_{n-1}\left(\beta_1,\dots,\beta_{n-1};-\left(\frac{t}{n}\right)^n(1-\xi_1)\dots(1-\xi_{n-1})\gamma\right)\prod_{i=1}^{n-1}\xi_i^{b_i-1}(1-\xi_i)^{\beta_i-1},$$

$$A_n^{-1}=\prod_{i=1}^{n-1}B(b_i,\beta_i); \qquad {}_0F_q(\mu_1,\dots,\mu_q;t)=$$

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$$= \sum_{s=0}^{\infty} \frac{\Gamma(\mu_1+1)\dots\Gamma(\mu_q+1)}{\Gamma(\mu_1+s+1)\dots\Gamma(\mu_q+s+1)} \frac{t^s}{s!};$$

$$\operatorname{Re}(b_i+\beta_i) > \operatorname{Re} b_i > 0^* \qquad (i=1,\dots,n-1);$$

$$u(t,x;b_1+\beta_1,\dots,b_l+\beta_l,b_{l+1},\dots,b_{n-1},c;f) = \tag{7^1}$$

$$= A_{l+1}\int_0^1 \dots \int_0^1 \prod_{i=1}^{l} \xi_i^{b_i-1}(1-\xi_i)^{\beta_i-1} u\left(t\sqrt[n]{\xi_1\dots\xi_l}, x; b_1,\dots,b_{n-1},c;f\right) \times$$

$$\times d\xi_1\dots d\xi_l, \qquad l=1,\dots,n-1; \tag{8}$$

$$\operatorname{Re}(b_i+\beta_i) > \operatorname{Re} b_i > 0, \qquad i=1,\dots,l; \tag{8^1}$$

$$u(t,x;b_1,\dots,b_{n-1},c;f) = \tag{9}$$

$$= Bt^{n(1-b_{n-1})}\left\{\prod_{i=1}^{n-1}\left[\left(\frac{\partial}{\partial t^n}\right)^{N_i} t^{n(b_i-b_{i-1}+N_i)}\right]\right\} u(t,x;b_1+N_1,\dots,b_{n-1}+N_{n-1},c;f);$$

$$B^{-1} = \prod_{i=1}^{n-1}(b_i,N_i), \qquad (b,N) = b(b+1)\dots(b+N-1); \quad N_i \geq 0, \text{ integers};$$

$$b_0 = 1; \qquad b_i \neq 0,-1,\dots,-(N_i-1) \quad (i=1,\dots,n-1); \tag{9^1}$$

here it is assumed that the linear operator X, acting with respect to the variables x, is commutable with integration with respect to the variables ξ (independent of x) in the right-hand sides of formulas (7), (8), and, respectively, with differentiation with respect to t in (9).

3.1. It may be practically useful to reduce problem (3)–(4) with coefficients variable in t to the corresponding problem (3)–(4) with coefficients constant in t (i.e., to the case when $k_1 = \dots = k_{n-1} = 0$). In this connection, we note that, since

$$(\delta+\alpha)t^l w(t,x) = t^l(\delta+\alpha+l)w(t,x) \tag{*}$$

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and $(\delta-(n-1))\ldots(\delta-1)\delta w = t^n D^n w$, equation $(3')$ for $b_i = i/n$ $(i = 1,\ldots,n-1)$ turns into $(D^n + c)u = Xu$. More generally: to the system of parameter values $k_1, k_2 = \ldots = k_{n-1} = 0$ in (3) there corresponds the system of parameters $b_1 = (k_1+1)/n,\ b_2 = 2/n,\ldots,\ b_{n-1} = (n-1)/n$ in $(3')$.**

3.2. In order to express $\tilde{u}(t,x;k'_1,\ldots,k'_{n-1},c';f)$ in terms of $\tilde{u}(t,x;k_1,\ldots,k_{n-1},c;f)$, one should, from system (5), using the values $k'_1,\ldots,k'_{n-1}$ and $k_1,\ldots,k_{n-1}$, find the corresponding $b'_1,\ldots,b'_{n-1}$ and $b_1,\ldots,b_{n-1}$. In the cases $\operatorname{Re} b'_i > \operatorname{Re} b_i > 0$ $(i = 1,\ldots,n-1)$, or $b'_i = b_i - N_i$ $(i = 1,\ldots,n-1)$, N_i are integers ≥ 0, $b'_i \neq 0,-1,-2,\ldots$, we achieve the goal by applying (7) or, respectively, (9). In all other cases, but under the conditions $\operatorname{Re} b_i > 0$ and $b'_i \neq 0,-1,-2,\ldots$ $(i = 1,\ldots,n-1)$, by successive use of (9) and (7), with appropriate N_i and β_i, we obtain the required reduction.*** In particular, the formula of transition from $u(c') \equiv u(t,x;b_1,\ldots,b_{n-1},c';f)$ to $u(c)$ can be obtained successively—

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* For uniqueness of the solution of problem $(3')$–(4), it is necessary that $\operatorname{Re} b_i \geq 1/n,\ i = 1,\ldots,n-1$ (see items 3, 4).

** The parameters $b_1,\ldots,b_{n-1}$ for arbitrarily prescribed $k_1,\ldots,k_{n-1}$ are the roots of an algebraic equation of degree $(n-1)$, obtained directly from system (5); among them, even for real $k_1,\ldots,k_{n-1}$, complex ones may occur.

*** If one of the parameters b_i (say b_1) is equal to $-r$, r an integer ≥ 0, then, using (9), with $N_1 = r-1$, we can reduce u for $b_1 = -r$ to the function u for $b_1 = -1$, and the latter (by a formula analogous to (1_4) from (9)) can be brought to u for $b_1 > 0$.

by applying formula (9) successively to $u(c')$ with $N_1 = \cdots = N_{n-1} = 1$, and formula (7) with $\beta_1 = \cdots = \beta_{n-1} = 1,\ \gamma = c' - c$. The formula thus obtained, when written for $b_i = i/n$ $(i = 1,\ldots,n-1)$, establishes a connection between the solutions of the Cauchy problem (4) for the equations $(D^n + c)u = Xu$ and $D^n u = Xu,\ n \geqslant 2$. For $n = 2$ it was obtained in [12]; see also formula (1_3) in [9]; compare in this connection [13], since in the case under consideration $(k_1 = \cdots = k_{n-1} = 0)$ the singularity of the problem disappears.

3.3. For $\gamma = 0$ formula (7) turns into formula (8), but with $l = n-1$. Formula (9) is obtained by inverting (8) for integer β_i. Applying (8) for $l = 1$, in view of Sec. 3.1, we obtain a formula expressing $\tilde{u}(t,x;k_1,0,\ldots,0,c;f)$, $k_1 > 0$, in terms of $\tilde{u}(t,x;0,0,\ldots,0,c;f)$, noted (in a special case) in [10]; see also [11]. For $n = 2$ formulas (7)–(9) and (12) were obtained in [9].

3.4. Putting in equation $(3')$ $u = t^l v$ and taking into account the equality (*) (Sec. 3.1) for $l = (1-b_i)n$ $(i = 1,\ldots,n-1)$, we find that, along with $u_0 \equiv u(t,x;b_1,\ldots,b_{n-1},c;f)$, the equation $(3')$ is also satisfied by

$$u_i \equiv t^{n(1-b_i)} u(t,x;b_1-b_i+1,\ldots,b_{i-1}-b_i+1,2-b_i,b_{i+1}-b_i+1,\ldots,b_{n-1}-b_i+1,c;f)*$$

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$(i = 1, \dots, n-1)$; hence it follows that if a solution of problem $(3')$–(4) exists for arbitrary values of b_i $(i = 1, \dots, n-1)$, then it is not unique if at least one of the parameters b_i $(i = 1, \dots, n-1)$, for example b_j, is such that $\operatorname{Re} b_j < 1/n$, for in that case u_j is a solution of equation $(3')$ satisfying zero initial conditions.

3.5. Formula (8) can be represented in the form

$$u(t, x; b_1 + \beta_1, \dots, b_l + \beta_l, b_{l+1}, \dots, b_{n-1}, c; f) = \tag{8'}$$

$$= \int_0^1 K_l(\xi; b_1, \dots, b_l; \beta_1, \dots, \beta_l)\, u(t \sqrt[n]{\xi}, x; b_1, \dots, b_{n-1}, c; f)\, d\xi$$

$$(l = 1, \dots, n-1)$$

with kernel obtained recurrently from

$$K_1(\xi; b_1; \beta_1) = \frac{1}{B(b_1, \beta_1)} \xi^{b_1 - 1} (1 - \xi)^{\beta_1 - 1}$$

by the formula

$$K_p(\xi; b_1, \dots, b_p; \beta_1, \dots, \beta_p) = \tag{10}$$

$$= \int_\xi^1 K_1(z; b_p; \beta_p)\, K_{p-1}\left(\frac{\xi}{z}; b_1, \dots, b_{p-1}; \beta_1, \dots, \beta_{p-1}\right) \frac{dz}{z} \quad (p = 2, \dots, n-1).$$

4. Let $w(t, x; f)$ be the solution of the problem

$$\partial w / \partial t = X w, \qquad w|_{t=0} = f(x), \tag{11}$$

and let $u(t, x; b_1, \dots, b_{n-1}; f)$ be the solution of problem $(3')$–(4) for $c = 0$. The formula

$$w(t, x; f) = \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{n-1} \frac{e^{-\xi_i} \xi_i^{b_i - 1}}{\Gamma(b_i)}\, u(n \sqrt[n]{\xi_1 \cdots \xi_{n-1}}\, t, x; b_1, \dots, b_{n-1}; f)\, d\xi_1 \cdots d\xi_{n-1}, \tag{12}$$

holds; here it is assumed that the linear operator X is interchangeable with integration with respect to $\xi_1, \dots, \xi_{n-1}$ in the right-hand side of (12).

Concerning formula (12), there is a statement analogous to that made in Sec. 3.5 with respect to formula (8).

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5. **Some applications of the preceding results.**
5.1. Let $X = \nabla_n$, $\nabla_n \equiv \partial^n/\partial x_1^n + \dots + \partial^n/\partial x_m^n$, and let $f(x)$ be an analytic function ($n > 2$). Put

$$s\{f\} = \frac{1}{n^m} \sum_{l_1,\dots,l_m=1}^{n} f(x_1 + t a_1 e^{2l_1\pi i/n}, \dots, x_m + t a_m e^{2l_m \pi i/n}), \tag{13}$$

* Thus we arrive, for equation (3′), at a generalization of one of the recurrence relations of Darboux–Weinstein [1,2]; the second generalization has the form

$$C_1 t^{1-n} D u(t, x; b_1, \dots, b_{n-1}, c; f) = u(t, x; b_1 + 1, \dots, b_{n-1} + 1, c; (X - c) f),$$

$$C_1 \equiv n^{n-1} b_1 \dots b_{n-1}.$$

where $a_j = (\eta_{1,j} \dots \eta_{n-1,j})^{2/n}$ ($j = 1, \dots, m$), and $\eta_{k,1}, \dots, \eta_{k,m}$ are the polar coordinates of a point of the sphere $S_{k,m}$ ($k = 1, \dots, n-1$) of radius 1 with center at the origin. For $m > 1$ consider the following averaging (14) (with weight) of the function $f(x)$:

$$V_n(t, x; f) = \frac{1}{\Omega} \int_{S^+_{1,m}} \cdots \int_{S^+_{n-1,m}} s\{f\} \prod_{j=1}^{n-1} (\eta_{j,1} \dots \eta_{j,m})^{(2j-n)/n} \, ds_{1,m} \cdots ds_{n-1,m}; \tag{14}$$

$$\Omega = 2^{(1-n)(m-1)} \prod_{j=1}^{n-1} \left[\Gamma\left(\frac{j}{n}\right)\right]^m \left[\Gamma\left(\frac{mj}{n}\right)\right]^{-1};$$

$S^+_{k,m}$ is the region of the sphere $S_{k,m}$ for which $\eta_{k,j} \geqslant 0$ ($j = 1, \dots, m$), and $dS_{k,m}$ is the surface element of the sphere $S_{k,m}$.

It can be shown that $V_n(t, x; f)$ satisfies the equation

$$t^{-n} \delta(\delta + m - n) \dots (\delta + (n-1)m - n) V_n = \nabla_n V_n \tag{15}$$

and the initial conditions (4)*. Consequently,

$$V_n(t, x; f) = u(t, x; m/n, \dots, (n-1)m/n, 0; f). \tag{16}$$

Thus, using (14), (16) for $m > 1$, we can, by formulas (7), (9) (see item 3.2), write the solution of problem (3)–(4), $X = \nabla_n$, explicitly (in the class of analytic

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initial data) for arbitrary $b_1, \dots, b_{n-1}, c$. Similarly also for $m = 1$; here it suffices to rely on the solution of problem (4) for the equation $D^n u = \partial^n u / \partial x_1^n$, namely:

$$u\left(t, x; \frac{1}{n}, \dots, \frac{n-1}{n}, 0; f\right) = \frac{1}{n}[f(x + \varepsilon_1 t) + \dots + f(x + \varepsilon_n t)], \qquad (17)$$

where $\varepsilon_1, \dots, \varepsilon_n$ are the roots of the n-th degree of 1.

5.2. Using formula (12) and taking into account (16) (and, for $m = 1$, (17)), we can also write the solution of problem (11) for $X = \nabla_n$ in explicit form (in the class of analytic initial data).

5.3. If we assume that the operator X is the operator of multiplication by the number λ, then

$$u(t, x; b_1, \dots, b_{n-1}, c; f) = {}_0F_{n-1}(b_1, \dots, b_{n-1}; (t/n)^n (\lambda - c))\, f(x)$$

and formula (7) turns into the “integral addition theorem”for the hypergeometric function ${}_0F_{n-1}(b_1, \dots, b_{n-1}; t)$, namely:

$${}_0F_{n-1}(b_1 + \beta_1, \dots, b_{n-1} + \beta_{n-1}; t_1 + t_2) =$$

$$= A_n \int_0^1 \cdots \int_0^1 {}_0F_{n-1}(b_1, \dots, b_{n-1}; \xi_1 \dots \xi_{n-1} t_1)\, {}_0F_{n-1}(\beta_1, \dots, \beta_{n-1}; (1 - \xi_1) \dots$$

$$\dots (1 - \xi_{n-1}) t_2) \prod_{i=1}^{n-1} \xi_i^{b_i - 1} (1 - \xi_i)^{\beta_i - 1}\, d\xi_i^{**}. \qquad (18)$$

Relation (18), as well as the relations obtained from formulas (8), (9), (12) under the assumption about the operator X made in item 5.3, being found independently, can be regarded as a heuristic source of formulas (7), (8), (9), (12); see in this connection ([15]).

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* For $n = 2$, (15) is the Darboux-Asgeirsson equation, and $V_2(t, x; f)$ is the mean value of the function $f(x)$ on the sphere of radius t with center at the point x.

** For $n = 2$ this is the well-known Sonine formula for Bessel functions ([16]).

Note: Figure translations are in progress. See original paper for figures.

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