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Abstract

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THE CLASSICAL FORMULATION OF THE QUESTION OF THE LOCALIZATION PRINCIPLE FOR FOURIER SERIES IN EIGENFUNCTIONS OF MULTIDIMENSIONAL DOMAINS

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A fundamental role in the theory of one-dimensional trigonometric Fourier series is played by Riemann' s famous localization principle, which asserts that *the convergence of the Fourier series of a function $f(x)$ at a given point depends on the behavior of the function $f(x)$ only in a neighborhood of that point* (see, for example, ([1]), p. 110). The following lemma plays the basic role in Riemann' s theory. **Lemma.** *If a function $f(x)$ belongs to the class L_1 on the whole segment $[0 \le x \le 2\pi]$ and vanishes on some segment $[a, b]$ contained in $[0, 2\pi]$, then the trigonometric Fourier series of the function $f(x)$ converges to zero uniformly on every segment lying strictly inside $[a, b]$.*

The problem of generalizing Riemann' s localization principle and the indicated basic lemma to the case of multiple trigonometric Fourier series arose long ago. However, for many years no substantial progress in this direction was obtained. Only the example of L. Tonelli ([2]) is known, which showed that there exists even a continuous function $f(x, y)$ in the closed square $R = [0 \le x \le 2\pi] \times [0 \le y \le 2\pi]$, vanishing in some square D contained in R, and nevertheless possessing a double Fourier series divergent at interior points of the square D. Tonelli' s example indicates that the question of generalizing the localization principle to the case of multiple Fourier series is quite nontrivial.

In the present work we consider Fourier series in eigenfunctions of the Laplace operator in an arbitrary N-dimensional domain g and with arbitrary boundary conditions ensuring the existence of a countable set of nonnegative eigenvalues $\{\lambda_n\}$ and a complete orthonormal system of eigenfunctions $\{u_n(x)\}$. Particular cases of such a consideration are multiple trigonometric Fourier series*, Fourier series in eigenfunctions of the N-dimensional ball, and, in general, of an N-dimensional bounded domain g with homogeneous boundary conditions of any of the three kinds.

For the Fourier series under consideration we give an exhaustive solution of the question of the localization principle in its classical formulation.

1°. Formulation of the results. The first result of the present work is a negative example establishing what minimal smoothness requirements must be imposed on the function f at points "far" from the point under consideration x_0 in order that convergence of the Fourier series of this function at the point x_0 may occur.

For $N \geq 2$ we consider the eigenfunctions of the N-dimensional ball Ω, corresponding to a homogeneous boundary condition of the first or second kind, and prove the existence of a function f satisfying the following

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* Corresponding to the N-dimensional rectangular parallelepiped and to boundary conditions ensuring the periodic continuation of the eigenfunctions through all faces.

requirements: 1) the function f and all its derivatives up to order $[N/2]-1$* are continuous in the closed ball Ω; 2) the function f vanishes, first, in some N-dimensional ball Ω_1 contained in Ω, and, second, in some strip near the boundary of the ball Ω; 3) the Fourier series of the function f in the eigenfunctions of the ball Ω does not converge at the center x_0 of the ball Ω_1 under any ordering of its terms (the n-th term of the indicated Fourier series does not tend to zero at the point x_0 as $n \to \infty$).

The example constructed by us shows that the existence, for the function f at points "far"from the point x_0 under consideration, even of continuous derivatives of order $[N/2]-1$ is still insufficient to ensure convergence of the Fourier series of this function at the point x_0 (despite the fact that in a neighborhood of the point x_0 this function is arbitrarily smooth, and on the boundary of the domain not only the function itself, but also all its derivatives, satisfy the corresponding boundary condition!).

A natural question arises: what will happen if we require that, at points "far" from the point x_0 under consideration, the function f possess derivatives of one higher order, i.e. of order $[N/2]$?

As an answer to this question we shall formulate a positive assertion which, in a certain sense, is a best possible generalization of Riemann' s basic lemma**.

Theorem. Let g be an arbitrary N-dimensional domain, and let f be an arbitrary function satisfying the following requirements: 1) throughout the domain g the function f belongs to the class $W_2^{[N/2]}(g)$; 2) in some domain D contained in g, the function f vanishes; 3) on the boundary of the domain g the function f satisfies conditions ensuring convergence of the series

$$\sum_{k=1}^{\infty} f_k^2 \lambda_k^{[N/2]} *** . \quad (1)$$

Then the Fourier series of the function f in the eigenfunctions of the domain g

converges, when summed in the order of increasing eigenvalues, to zero uniformly in every strictly interior subdomain D' of the domain D.

2°. We shall indicate the scheme for deriving the results formulated above. First of all, let us note that the formulated theorem is in fact proved in paper (3) (see Remark 9 on p. 100 of the cited paper). It should only be noted that, although in paper (3) the eigenfunctions of three homogeneous boundary-value problems are discussed, all the results of that paper, under the assumption of convergence of the series (1), are valid for the arbitrary boundary conditions considered in the present paper.

Thus, it remains for us to outline the scheme for constructing the negative example indicated above. Let Ω be the N-dimensional ball of radius R, $0 < R_1 < R_2 < R_3 < R_4 < R$. Denote by the symbol $u_n(r)$ the normalized eigenfunctions of the ball Ω possessing spherical symmetry, and take into account that

$$u_n(0) = C_n \lambda_n^{(N-1)/4}, \tag{2}$$

where $\lim_{n\to\infty} |C_n| = C > 0$ (for both the first and the second boundary-value problems).

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* Here and below square brackets denote that the integer part of the number enclosed in them is taken.

** The best possible character of the formulated theorem follows from comparing it with the example indicated above.

*** For convergence of the series (1) in the case of eigenfunctions of any of the three homogeneous boundary-value problems, it is sufficient that the functions $f, \Delta f, \ldots, \Delta^m f$ (where $m = [(N-2)/4]$ for the first boundary-value problem and $m = [(N-4)/4]$ for the second and third boundary-value problems) satisfy, in the generalized sense, the corresponding boundary condition (see Lemma 1 on p. 93 of paper (3)).

As in § 9 of work (3), we shall rely on the following lemma, due to Banach:

If $v_n(r)$ is a sequence of functions summable on the segment $a \leqslant r \leqslant b$ such that the numerical sequence $\int_a^b |v_n(r)|\, d$ is unbounded, then there exists a function $\bar{f}(r)$, continuous on the segment $a \leqslant r \leqslant b$, such that the numerical sequence $\int_a^b \bar{f}(r) v_n(r)\, dr$ is also unbounded.

We shall apply this lemma to the sequence $v_n(r) = u_n(r) r^{N-1} \ln \lambda_n$, putting $a = R_2$, $b = R_3$. In this way we establish the existence of a function $\bar{f}(r)$, continuous on the segment $R_2 \leqslant r \leqslant R_3$, such that

$$\int_{R_2}^{R_3} \bar{f}(r)u_n(r)\,dr = \frac{A_n}{\ln\lambda_n}, \tag{3}$$

where $\{A_n\}$ is some unbounded sequence. Further, for simplicity, we shall consider the case when the number $[N/2]$ is odd. Define the desired function $f(r)$ in the annulus $R_2 \leqslant r \leqslant R_3$ as the solution of the equation

$$\Delta^{[(N-2)/4]} f = (-1)^{[(N-2)/4]}\bar{f}(r)$$

with boundary conditions ensuring that, on the boundary of the indicated annulus, the functions $f, \Delta f, \dots, \Delta^{[(N-6)/4]} f$ vanish. In the annulus $R_1 \leqslant r \leqslant R_2$ we put the function $f(r)$ equal to the polynomial

$$f_1(r) = a_0 + a_1 r^2 + a_2 r^4 + \dots + a_{2[N/2]-1} r^{4[N/2]-2}, \tag{4}$$

in which the constants $a_0, a_1, a_2, \dots, a_{2[N/2]-1}$ are chosen so that the function $f_1(r)$ and all its derivatives up to order $[N/2]-1$ vanish at $r = R_1$, and at $r = R_2$ coincide with the corresponding derivatives of the function $f(r)$.

In a completely analogous way the function $f(r)$ is constructed in the annulus $R_3 \leqslant r \leqslant R_4$, where we take it equal to a certain polynomial $f_2(r)$, which, together with its derivatives up to order $[N/2]-1$, vanishes at $r = R_4$ and coincides with $f(r)$ and its derivatives at $r = R_3$. In the ball $0 \leqslant r \leqslant R_1$ and in the annulus $R_4 \leqslant r \leqslant R$ we set the function $f(r)$ equal to zero.

With this construction, the function $f(r)$ will: 1) possess continuous derivatives up to order $[N/2]-1$ in the whole closed ball Ω; 2) vanish in the ball $\Omega_1 = \{r \leqslant R_1\}$; 3) vanish in the annulus $R_4 \leqslant r \leqslant R$, which is a boundary strip of the ball Ω. It remains to verify that the n-th term of the Fourier series of the function $f(r)$ does not tend to zero at the point $r = 0$ as $n \to \infty$. By construction,

$$\Delta^{[(N-2)/4]} f = \begin{cases} (-1)^{[(N-2)/4]}\bar{f}(r), & \text{for } R_2 < r < R_3, \\ \Delta^{[(N-2)/4]} f_1(r), & \text{for } R_1 < r < R_2, \\ \Delta^{[(N-2)/4]} f_2(r), & \text{for } R_3 < r < R_4, \\ 0, & \text{for } r < R_1 \text{ and } R_4 < r < R, \end{cases} \tag{5}$$

and for the Fourier coefficients the relation

$$\{\Delta^{[(N-2)/4]} f\}_n = (-1)^{[(N-2)/4]} \lambda_n^{[(N-2)/4]} f_n \tag{6}$$

is valid. From relations (5) and (6) we conclude that

$$f_n = \frac{\omega_N}{\lambda_n^{[(N-2)/4]}} \left\{ \int_{R_2}^{R_3} \bar{f}(r)u_n(r)\,dr + (-1)^{[(N-2)/4]} \int_{R_1}^{R_2} \Delta^{[(N-2)/4]} f_1(r)u_n(r)\,dr + \right.$$

$$+(-1)^{[(N-2)/4]}\int_{R_3}^{R_4}\Delta^{[(N-2)/4]}f_2(r)u_n(r)\,dr\Bigg\} \qquad \left(\omega_N=\frac{2(\sqrt{\pi})^N}{\Gamma(N/2)}\right). \qquad (7)$$

From the form of the functions $f_1(r)$ and $f_2(r)$ (see formula (4)), by elementary calculations we find that the last two integrals in formula (7) are of order $O(1/\sqrt{\lambda_n})$. But then, taking into account relations (2) and (3), we obtain that the n-th term of the Fourier series $f_n u_n(0)$ does not tend to zero as $n \to \infty$.

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Note: Figure translations are in progress. See original paper for figures.