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Abstract

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MATHEMATICS

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AN ESTIMATE OF THE EXTERNAL STABILITY NUMBER OF A GRAPH

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Only finite undirected graphs without loops or multiple edges are considered ([1]). A subset T of the vertices of a graph is called **externally stable** if every vertex not belonging to T is adjacent to at least one vertex of T. The least cardinality of an externally stable set of a graph G is denoted by $\beta(G)$ and is called the **external stability number** of the graph G. In the present note the relation between the external stability number and the number of edges in an n-vertex graph is studied. Analogous problems for the internal stability number and the chromatic number were solved in ([2–4]). We shall denote by $\sigma(x)$ the degree of the vertex x, and by $\sigma(G)$ the maximum degree of a vertex of the graph G.

Lemma 1. If G is an n-vertex graph, then $\sigma(G) \leq n - \beta(G)$.

Lemma 2. If G is a connected n-vertex graph, where $n \geq 2$, then $\beta(G) \leq [n/2]$.*

A graph G will be called **critical** if either it is complete, or if, upon joining by an edge two of its nonadjacent vertices, one obtains a graph whose external stability number is equal to $\beta(G) - 1$.

We shall call the **gluing** of vertices x and y of a graph G the operation of replacing them by one vertex adjacent to those and only those vertices which were adjacent to at least one of the vertices x and y.

Lemma 3. If a graph H is obtained from a critical graph G by gluing two nonadjacent vertices, then $\beta(H) = \beta(G) - 1$.

We shall denote by $f(n, k)$ the greatest number of edges that a connected n-vertex graph with external stability number k can have. In what follows, when we write $f(n, k)$, we shall, without special reservation, mean that $k \leq [n/2]$ for $n \geq 2$ (Lemma 2) and $k = 1$ for $n = 1$.

We shall denote by $C(n, k)$ a connected n-vertex graph with $f(n, k)$ edges and with $\beta(C(n, k)) = k$. Obviously, the graph $C(n, k)$ is critical.

Theorem 1.

$$f(n,1) = \frac{n(n-1)}{2}, \qquad f(n,2) = \frac{n(n-2)}{2},$$

$$f(n,k) = (n-k+1)(n-k)/2 \quad \text{for } k \geq 3.$$

Proof. $f(n,1) = n(n-1)/2$, since $C(n,1)$ is the complete n-vertex graph.

$f(n,2) \leq [n(n-2)/2]$, since, by Lemma 1, $\sigma(C(n,2)) \leq n-2$. On the other hand, for $n \geq 4$ it is easy to construct a connected n-vertex graph

* The brackets [ ] denote the integer part of a number.

with $[n(n-2)/2]$ edges and with external stability number 2; for even n such a graph is obtained by deleting from the complete n-vertex graph a matching consisting of $n/2$ edges; for odd n, by deleting from the complete n-vertex graph a matching of $(n-1)/2$ edges and one edge issuing from a vertex not incident with any edge of this matching. Consequently,

$$f(n,2) = [n(n-2)/2].$$

Let us prove the equality

$$f(n,k) = (n-k+1)(n-k)/2$$

for $k \geq 3$.

First,

$$f(n,k) \geq (n-k+1)(n-k)/2,$$

since for any natural n and k, $3 \leq k \leq [n/2]$, one can construct a connected n-vertex graph H with $(n-k+1)(n-k)/2$ edges and with $\beta(H) = k$: the vertices of H are

$$p_1, p_2, \dots, p_k, \; q_1, q_2, \dots, q_{n-k};$$

the vertex p_i is adjacent to q_i for $1 \leq i \leq k-1$; p_k is adjacent to all vertices q_l for $l \geq k$; the vertices $q_1, q_2, \dots, q_{n-k}$ are pairwise adjacent. Note that from the inequality just proved it follows easily that

$$\sigma(C(n,k)) \geq n-2k+2$$

for $k \geq 3$.

To complete the proof of the theorem it remains to show that

$$f(n,k) \leq (n-k+1)(n-k)/2$$

for $k \geq 3$. We prove this by induction on k, beginning with $k = 3$.

Let x be a vertex of maximum degree in the graph $C(n, 3)$; $\Gamma(x)$ is the set of vertices adjacent to x. Since $\sigma(x) \geq n-4$, by Lemma 1 only two cases are possible:

$$\sigma(x) = n - 4$$

and

$$\sigma(x) = n - 3.$$

Let $\sigma(x) = n-4$; let x_1, x_2, x_3 be vertices of the graph $C(n, 3)$ not adjacent to x. Since $\beta(C(n, 3)) = 3$, in Γx there is not a single vertex adjacent simultaneously to the vertices x_1, x_2, x_3, and none of the vertices x_1, x_2, x_3 is nonadjacent to the other two. Hence

$$\sigma(x_1) + \sigma(x_2) + \sigma(x_3) \leq 2(n-4) + 2 = 2(n-3).$$

Consequently, the number of edges of the graph $C(n, 3)$ is not greater than

$$2(n-3) + (n-3)(n-4)/2 = (n-2)(n-3)/2.$$

Now let $\sigma(x) = n-3$; x_1 and x_2 are vertices not adjacent to x. Denote by L_1 and L_2 the subsets of vertices of Γx adjacent respectively to x_1 and x_2. Obviously, x_1 and x_2 are nonadjacent; $|L_1| > 0$; $|L_2| > 0$*, $L_1 \cap L_2 = \emptyset$; whichever two vertices are taken—one from L_1 and the other from L_2—there is in Γx a vertex adjacent to neither of these two. Hence it is not hard to infer that

$$\sigma(x_1) + \sigma(x_2) = |L_1| + |L_2|$$

and that the subgraph generated by the vertices of $\Gamma(x)$ has no more than

$$(n-3)(n-4)/2 - |L_1| - |L_2|$$

edges. Therefore,

$$f(n, 3) \leq |L_1| + |L_2| + (n-3)(n-4)/2 - |L_1| - |L_2| + n - 3 = (n-2)(n-3)/2.$$

Suppose now that for all k, $3 \leq k < k_0$, it has been proved that

$$f(n, k) \leq (n-k+1)(n-k)/2,$$

and suppose, contrary to the assertion of the theorem, that

$$f(n, k_0) > (n-k_0+1)(n-k_0)/2.$$

Then in the graph $C(n, k_0)$, for any two nonadjacent vertices x' and x'', there is a third vertex adjacent to both of them, since otherwise, by "gluing" x' and x'', we would obtain a connected $(n-1)$-vertex graph with number of edges $> f(n-1, k_0-1)$, whose external stability number is equal to $k_0 - 1$, which contradicts the induction hypothesis.

Let y be a vertex of maximum degree in the graph $C(n, k_0)$, Γy the set of vertices adjacent to y; let $y_1, y_2, \dots, y_j$ be vertices not adjacent to y. Since

$$\sigma(C(n, k_0)) \geq n - 2k_0 + 2,$$

we have

$$j \leq 2k_0 - 3.$$

Consider two cases.

Case 1. $\sigma(y) \geq n - 2k_0 + 3$. Then $j \leq 2k_0 - 4$. Choose from the set of vertices

$$y_1, y_2, \dots, y_j$$

the maximum number of disjoint pairs, each of which consists of two nonadjacent vertices. Since $j \leq 2k_0 - 4$, the number of such pairs $r \leq k_0 - 2$. Let v_i be a vertex adjacent to both vertices of the i-th pair $(1 \leq i \leq r)$. If $r = k_0 - 2$, then the vertices

$$y,\ v_1,\ v_2, \dots, v_r$$

form an externally stable set of the graph $C(n, k_0)$, which contradicts

$$\beta(C(n, k_0)) = k_0.$$

Now let $r \leq k_0 - 3$. Since the vertices among $y_1, y_2, \dots, y_j$ that did not enter into the selected pairs are pairwise adjacent, then, taking one of them and the vertices

$$y,\ v_1,\ v_2, \dots, v_r,$$

we

* If A is a finite set, then by $|A|$ is denoted the number of its elements.

again obtain an externally stable set of the graph $C(n, k_0)$ having no more than $k_0 - 1$ vertices, which again is impossible.

Case 2. $\sigma(y) = n - 2k_0 + 2$. Then $j = 2k_0 - 3$. If in Γy there is a vertex adjacent to at least three vertices among $y_1, y_2, \dots, y_{2k_0-3}$, or if among the vertices $y_1, y_2, \dots, y_{2k_0-3}$ there is a vertex adjacent to at least two others, then we easily arrive at case 1. If the indicated possibilities are excluded, then

$$\sigma(y_1) + \sigma(y_2) + \dots + \sigma(y_{2k_0-3}) \leq 2(n - 2k_0 + 2) + 2k_0 - 4 = 2n - 2k_0.$$

Therefore the number of edges of the graph $C(n, k_0)$ must be no more than

$$\left[(n - 2k_0 + 3)(n - 2k_0 + 2) + 2n - 2k_0\right]/2 \leq (n - k_0 + 1)(n - k_0)/2,$$

which contradicts the assumption $f(n, k_0) > (n - k_0 + 1)(n - k_0)/2$.

Theorem 1 is proved.

Corollary 1. *If a connected n-vertex graph G has m edges $(m \geq n-1)$, then*

$$1 \leq \beta(G) \leq \begin{cases} \min\left\{\left[\frac{n}{2}\right], \left[\frac{1+2n-\sqrt{8m+1}}{2}\right]\right\}, & \text{if } m \leq \frac{(n-2)(n-3)}{2}, \\ \min\left\{\left[\frac{n^2-2m}{2}\right], 2\right\}, & \text{if } m > \frac{(n-2)(n-3)}{2}. \end{cases}$$

Let n and k be positive integers such that $k \leq n$. Denote by $g(n,k)$ the greatest number of edges in an n-vertex graph with external stability number k, and by $M(n,k)$ an n-vertex graph with $g(n,k)$ edges and with $\beta(M(n,k)) = k$.

Theorem 2.

$$g(n,k) = \begin{cases} \frac{n(n-1)}{2}, & \text{for } k = 1, \\ \left[\frac{(n-k+2)(n-k)}{2}\right], & \text{for } k \geq 2. \end{cases}$$

For fixed n and k, the graphs $M(n,k)$ are isomorphic.

Proof. The theorem is obvious for $k = 1$ and $k = 2$. By virtue of

$$g(n,k) \leq g(n+1, k+1)$$

we have

$$g(n,k) \geq g(n-k+2, 2) = \left[\frac{(n-k+2)(n-k)}{2}\right]$$

for $k \geq 2$. Hence it follows that for $k \geq 3$ the graph $M(n,k)$ is disconnected. Indeed, if $k > [n/2]$, then this follows from Lemma 2, while if $3 \leq k \leq [n/2]$, then from the inequality

$$g(n,k) \geq \left[\frac{(n-k+2)(n-k)}{2}\right] > (n-k+1)(n-k)/2 = f(n,k).$$

Therefore, for $k \geq 3$, the graph $M(n,k)$ cannot have a connected component whose external stability number is ≥ 3. Further, by a simple count of edges one can verify that the graph $M(n,k)$ does not have two connected components each of which differs from an isolated vertex. Consequently, for $k \geq 3$ the graph $M(n,k)$ consists of $k-2$ isolated vertices and a graph isomorphic to $M(n-k+2, 2)$. Theorem 2 is proved.

Corollary 2. *If an n-vertex graph H has m edges, then*

$$\max\{n-m, 1\} \leq \beta(H) \leq [\, n+1-\sqrt{1+2m}\,].$$

Let us note in conclusion that in the class of all n-vertex graphs with m edges (analogously in the class of n-vertex connected graphs with m edges $(m \geq n-1)$)

there exists a graph with external stability number equal to any preassigned integer lying within those limits indicated by Corollary 2 (respectively, Corollary 1).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.