



Soviet-era science, translated into English

PHYSICS

B. V. ELISEEV

1965

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196501.70619>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

PHYSICS

B. V. ELISEEV

DECAY OF HOMOGENEOUS TURBULENCE IN A WEAKLY CONDUCTING LIQUID

(Presented by Academician M. A. Leontovich on 24 X 1964)

The study of turbulent motion of a liquid in the case where the nonlinear terms in the Navier–Stokes equation are negligibly small was begun in the works of M. D. Millionshchikov ⁽¹⁾ and L. G. Loitsyanskii ⁽²⁾. M. D. Millionshchikov investigated the general solution for even correlations, and for a special case of initial data found the law of decrease of the energy of turbulent motion. Later, in ⁽³⁾, it was shown that this law is asymptotically valid for an arbitrary initial distribution. In the present work an analogous law is found for the decay of homogeneous turbulence in a weakly conducting liquid situated in a magnetic field.

Let us consider such a motion of a conducting incompressible liquid in a magnetic field which may be regarded as homogeneous in Taylor' s sense ⁽⁴⁾, and for which the mean velocity is equal to zero. Taking the characteristic magnetic Reynolds number to be less than unity, we write the Navier–Stokes equation of motion as

$$\frac{\partial v_i}{\partial t} = -\frac{\partial P}{\partial x_i} \frac{1}{\rho} + \nu \frac{\partial^2 v_i}{\partial x_k^2} + \frac{f_i}{\rho},$$

$$f_i = \frac{[\mathbf{jB}]_i}{c}, \quad j_i = \sigma \frac{[\mathbf{vB}]_i}{c}, \quad \frac{\partial v_i}{\partial x_i} = 0. \quad (1)$$

Here σ is the conductivity, and \mathbf{B} is the external magnetic field.

From (1), by the usual method ⁽⁴⁾, we obtain the equation for the second moments of the velocities

$$\frac{\partial}{\partial t} Q_{ij} = P_{ij} + 2\nu \nabla^2 Q_{ij} + F_{ij},$$

$$Q_{ij} = \overline{v_i v_j'}, \quad P_{ij} = \frac{1}{\rho} \left(\frac{\partial}{\partial r_i} \overline{P v_j'} - \frac{\partial}{\partial r_j} \overline{P v_i'} \right), \quad F_{ij} = \frac{\overline{f_i v_j'} + \overline{f_j v_i'}}{\rho}; \quad (2)$$

Q_{ij} is the tensor determining the correlation between the velocity components $v_i(x)$ at the point P and the velocity components $v'_j(x')$ at the point P' . It is assumed that all quantities in (2) depend only on the difference $\mathbf{r} = \mathbf{x}' - \mathbf{x}$.

For F_{ij} we have the expression

$$F_{ij} = \frac{1}{\tau_m} (-2Q_{ij} + x_i^1 x_\alpha^1 Q_{\alpha j} + x_j^1 x_\alpha^1 Q_{i\alpha}), \quad \tau_m = \frac{\rho c^2}{\sigma B^2}, \quad \mathbf{x}^1 = \frac{\mathbf{B}}{B}. \quad (3)$$

Let the unit vectors $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$ form a rectangular coordinate system. An arbitrary tensor of the second rank in this system can then be written in the form

$$Q_{ij} = Q\delta_{ij} + a^{\alpha\beta} x_i^\alpha x_j^\beta + b^\beta \varepsilon_{ijk} x_k^\beta; \quad (4)$$

$$x_i^\alpha = 1, \quad \alpha = i; \quad x_i^\alpha = 0, \quad \alpha \neq i,$$

where the functions $Q, a^{\alpha\beta}, b^\beta$ depend on $r^\alpha = (\mathbf{r} \cdot \mathbf{x}_\alpha)$. In the presence of a magnetic field the tensor Q_{ij} is axisymmetrical. It possesses reflection symmetry and, consequently, the functions b^β are equal to zero.

The incompressibility condition leads to the equations $\partial Q_{ij}/\partial r_i = \partial Q_{ij}/\partial r_j = 0$, whence it follows that

$$Q_\alpha + a_\beta^{\alpha\beta} = 0, \quad \bar{Q}_\alpha + a_\beta^{\beta\alpha} = 0, \quad (5)$$

where $Q_\alpha = \partial Q/\partial r^\alpha$, $a_\gamma^{\alpha\beta} = \partial a^{\alpha\beta}/\partial r^\gamma$.

The properties of the tensor Q_{ij} , in addition to the solenoidality condition, are also determined by the quasi-neutrality condition $\text{div } j = 0$.

It leads to the equations:

$$\frac{\partial}{\partial r_j} \varepsilon_{i\alpha\beta} x_\beta^1 Q_{i\alpha} = 0, \quad \frac{\partial}{\partial r_j} \varepsilon_{i\alpha\beta} x_\beta^1 Q_{\alpha i} = 0$$

or

$$[\mathbf{x}^1 \mathbf{x}^\alpha]_i Q_\alpha + a_2^{\alpha 3} x_i^\alpha - a_3^{2\alpha} x_i^\alpha = [\mathbf{x}^1 \cdot \mathbf{x}^\alpha]_i Q_\alpha + a_2^{3\alpha} x_i^\alpha - a_3^{2\alpha} x_i^\alpha = 0, \quad (6)$$

taking (4) into account, the tensor F_{ij} takes the form

$$F_{ij} = -\frac{1}{\tau_m} [2Q(\delta_{ij} - x_i^1 x_j^1) + 2a^{\alpha\beta} x_i^\alpha x_j^\beta - x_i^1 x_j^\alpha a^{1\alpha} - x_j^1 x_i^\alpha a^{\alpha 1}].$$

Let us next consider the tensor $P_{ij} = \partial L_j(\mathbf{r})/\partial r_i - \partial L_i(-\mathbf{r})/\partial r_j$, where $L_i = \frac{1}{\rho} P v'_i$. If the vector L_i has the representation $L_i = l^\alpha x_i^\alpha$, where l^α are scalar functions of r^α , then

$$P_{ij} = l^\beta x_i^\alpha x_j^\beta - \tilde{l}_\beta^\alpha x^\alpha x_j^\beta, \quad \tilde{l}^\alpha(r^\alpha) = l^\alpha(-r^\alpha).$$

The incompressibility and quasi-neutrality conditions lead to equations which the functions l^α and \tilde{l}^α must satisfy:

$$\varepsilon_{i\alpha\beta} x_\beta^1 \frac{\partial L_\alpha}{\partial r_i} = 0; \quad \frac{\partial L_i(\mathbf{r})}{\partial r_i} = 0, \quad \frac{\partial}{\partial r_j} \left(P_{ij} + \frac{x_j^1 x_\alpha^1 Q_{i\alpha}}{\tau_m} \right) = 0,$$

$$\varepsilon_{i\alpha\beta} x_\beta^1 \frac{\partial L_\alpha(-\mathbf{r})}{\partial r_i} = 0, \quad \frac{\partial L_i(-\mathbf{r})}{\partial r_i} = 0, \quad \frac{\partial}{\partial r_i} \left(P_{ij} + \frac{x_i^1 x_\alpha^1 Q_{\alpha j}}{\tau_m} \right) = 0$$

or

$$l_\alpha^\alpha = \tilde{l}_\alpha^\alpha = 0, \quad l_2^3 = l_3^2, \quad \tilde{l}_2^3 = \tilde{l}_3^2, \quad (7)$$

$$x_i^\alpha l_{\beta\beta}^\alpha + \frac{1}{\tau_m} (Q_1 x_i^1 + a_1^{\alpha 1} x_i^\alpha) = 0, \quad -x_i^\alpha \tilde{l}_{\beta\beta}^\alpha + \frac{1}{\tau_m} (Q_1 x_i^1 + a_1^{\alpha 1} x_i^\alpha) = 0.$$

From (2) it follows that the scalar functions Q , $a^{\alpha\beta}$ satisfy the system

$$D(Q + a^{11}) = l_1^1 - \tilde{l}_1^1,$$

$$D(Q + a^{22}) = -\frac{2(Q + a^{22})}{\tau_m} + l_2^2 - \tilde{l}_2^2,$$

$$D(Q + a^{33}) = -\frac{2(Q + a^{33})}{\tau_m} + l_3^3 - \tilde{l}_3^3;$$

$$D a^{\alpha\beta} = -2 \frac{a^{\alpha\beta}}{\tau_m} + l_\alpha^\beta - \tilde{l}_\beta^\alpha \quad (\alpha \neq 1, \alpha \neq \beta, \beta \neq 1), \quad (8)$$

$$D a^{\alpha 1} = -\frac{a^{\alpha 1}}{\tau_m} + l_\alpha^1 - \tilde{l}_1^\alpha \quad (\alpha \neq 1),$$

$$D\psi = \frac{\partial\psi}{\partial t} - 2\nu \left[\frac{\partial^2}{(\partial r^1)^2} + \frac{\partial^2}{(\partial r^2)^2} + \frac{\partial^2}{(\partial r^3)^2} \right] \psi,$$

where ψ is an arbitrary scalar function.

Introduce the dimensionless variables $\bar{t} = t/\tau_m$, $\bar{r} = r/\sqrt{\nu\tau_m}$, $\bar{a}^{\alpha\beta} = (\nu/\tau_m)a^{\alpha\beta}$, $\bar{l}^i = \sqrt{\nu/\tau_m}l^i$ and apply the Fourier integral transform to system (8) and to equations (5), (6), (7). Then we obtain that the equations for the Fourier images of any of the functions $Q + a^{11}$, $Q + a^{22}$, $Q + a^{33}$, $a^{\alpha\beta}$ ($\alpha \neq \beta$) have exactly the same form:

$$\partial\bar{\psi}/\partial\bar{t} = -2(k^2 + k_1^2/k^2)\bar{\psi}, \quad (9)$$

where $\bar{\psi}$ is the Fourier transform of the function ψ , $k^2 = k_1^2 + k_2^2 + k_3^2$.

Integrating (9) and passing to the functions ψ themselves, we obtain (in dimensionless variables)

$$\psi(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \psi(\xi, 0) d\xi \int_{-\infty}^{\infty} \exp \left[-2 \left(k^2 + \frac{k_1^2}{k^2} \right) + i\mathbf{k}(\mathbf{r} - \xi) \right] d\mathbf{k}. \quad (10)$$

The second integral in (10) can be transformed as follows:

$$\begin{aligned} & 8 \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \exp\{-2k^2 t\} \prod_{\alpha=1}^3 \cos k_{\alpha} \rho_{\alpha} dk_1 dk_2 dk_3 = \\ & = \int_0^{\infty} dk \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\varphi \sum_{\alpha=1}^8 \exp[-2k^2 t - 2t \sin^2 \theta \cos^2 \varphi + ik f_{\alpha}] k^2 \sin \theta = \\ & = \frac{\sqrt{\pi}}{8\sqrt{2}} \frac{1}{t^{3/2}} \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\varphi \sin \theta \sum_{\alpha=1}^8 \Phi \left(\frac{3}{2}, \frac{1}{2}, -\frac{f_{\alpha}^2}{8t} \right) \exp[-2t \sin^2 \theta \cos^2 \varphi]. \end{aligned} \quad (11)$$

In (11), Φ is the confluent hypergeometric function,

$$\Phi(\alpha, \beta, z) = 1 + \frac{\alpha z}{\beta 1!} + \frac{\alpha(\alpha+1) z^2}{\beta(\beta+1) 2!} + \dots, \quad \rho = \mathbf{r} - \xi, \quad \rho_1 = \rho \sin \theta' \cos \varphi', \dots,$$

and the functions f_i do not depend on time.

Let us expand the function Φ in a series and retain only the first two terms as $t \rightarrow \infty$. The first term of the expansion vanishes upon integration over ξ , as was shown in (3). The second term of the expansion gives the decay law $\psi \sim t^{-5/2}$ if in (11) $\tau_m \rightarrow \infty$ (i.e., when the exponential factor in (11) is equal to unity), and for finite τ_m , $\psi \approx t^{-3}$. Indeed, after expanding Φ in (11), integration with respect to φ gives a Bessel function of imaginary argument of zero or first order. Using its asymptotic expansion, one can show that

$$\psi = -\frac{3\sqrt{\tau_m}}{2^{11}\pi t^3\sqrt{\cdot}} \int_{-\infty}^{\infty} |\mathbf{r} - \xi|^2 \psi(\xi, 0) (2 \cos^2 \theta' + \sin^2 \theta' \sin^2 \varphi') d\xi,$$

whence the asymptotic behavior of the function ψ as t^{-3} follows.

If the influence of pressure is neglected, then the correlations between velocity components perpendicular to the magnetic field would decrease with time as $\exp[-2t/\tau_m]$; correlations between velocity components, one of which is parallel and the other perpendicular to the magnetic field, would decrease as $\exp[-t/\tau_m]$ (if the influence of viscosity is neglected). In this case the correlations between velocity components parallel to the magnetic field would decrease only under the influence of viscosity. However, the action of pressure leads to all correlations decreasing in time in the same way, according to the law t^{-3} . Turbulent motions along the field create disturbances perpendicular to the field, and their energy decreases more rapidly than it would as a result of viscous dissipation alone. The asymptotic law of degeneration of turbulence, as in hydrodynamics, turns out to be universal (independent of the initial conditions).

The author expresses deep gratitude to A. A. Vedenov, under whose supervision this work was carried out.

Received
23 X 1964

CITED LITERATURE

1. M. D. Millionchikov, DAN, **22**, 236 (1940).
2. L. G. Loitsyanskii, Tr. TsAGI, No. 440 (1939).
3. G. K. Batchelor, A. A. Townsend, Proc. Roy. Soc., A **194**, No. 1039, 527 (1948).
4. G. Batchelor, *The Theory of Homogeneous Turbulence*, IL, Moscow, 1955.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.