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Abstract

Full Text

Mathematics

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Lebesgue Measure of Subsets of Euclidean Space as the Highest Variation of the Distance Function to a Closed Set*

(Presented by Academician M. A. Lavrent'ev on 24 VIII 1964)

Let G be a bounded open set in the Euclidean plane R_2 . At each point $z \in R_2$ with coordinates x, y , the function $r(x, y) = r(z) = r(z, R_2 \setminus G)$ is defined, where r is distance in R_2 . The set $E_t = \{z : r(z) = t\}$ is called the t -level of the function $r(z)$.

Theorem 1. For almost all $t \in (0, T]$, where $T = \max_{z \in G} r(z)$, each component E_t^α of the set E_t is a closed rectifiable Jordan curve (whose length l_t^α is in the usual sense) or a point; the length l_t of the set E_t (equal to $\sum_\alpha l_t^\alpha$) as a function of t is summable on the interval $(0, T]$, and the planar Lebesgue measure

$$\text{mes}_2 G = \int_0^T l_t dt. \quad (1)$$

Proof. Since $|r(z') - r(z'')| \leq r(z', z'')$ for any points $z', z'' \in R_2$ ((³, p. 373), the function $r(z)$ satisfies a Lipschitz condition with constant 1. By a known theorem of V. V. Stepanov ((⁴, p. 449), at almost every point the function $r(z)$ has a total differential. Obviously, at every such point $z_1 \in G$, $|\text{grad } r(z_1)| \leq 1$.

Let $z \in R_2 \setminus G$ be such that $r(z_1) = r(z_1, z)$, and let z_2 lie on the segment $\overline{z_1 z}$. It is easy to show that $r(z_1) - r(z_2) = r(z_1, z_2)$. Therefore for the function $-r(z)$ the direction of the gradient coincides with the direction of the vector $\overline{z_1 z}$, whence it follows that almost everywhere in G

$$|\text{grad } r(z)| = |\text{grad}[-r(z)]| = 1. \quad (2)$$

The Steklov means

$$r_h(z) = r_h(x, y) = \frac{1}{h^2} \int_0^h \int_0^h r(x + \xi, y + \eta) d\xi d\eta$$

of the function $r(z)$, absolutely continuous in each of the arguments x, y , have the following well-known properties:

- 1) at every point $z \in R_2$, the function $r_h(x, y)$ has continuous partial derivatives with respect to x and to y ;
- 2) $\partial r_h / \partial \sigma = [\partial r / \partial \sigma]_h$ at points of differentiability of $r(z)$ (here $[\partial r / \partial \sigma]_h$ is the Steklov mean of the derivative $\partial r / \partial \sigma$; $\sigma = x, y$);

* The present note is adjacent to the works ^(1,2); we use some concepts and theorems from these works.

- 3) as $h \rightarrow 0$, $r_h(z) \rightarrow r(z)$ uniformly on every bounded closed set $F \subset R_2$.

Applying the Cauchy-Bunyakovsky inequality, we obtain, for every point $z \in R_2$ at which $r(z)$ is differentiable,

$$|\text{grad } r_h(z)|^2 = \left[\frac{\partial r}{\partial x} \right]_h^2 + \left[\frac{\partial r}{\partial y} \right]_h^2 \leq \frac{1}{h^2} \int_0^h \int_0^h |\text{grad } r(x + \xi, y + \eta)|^2 d\xi d\eta \leq 1,$$

i.e. $|\text{grad } r_h(z)| \leq 1$ almost everywhere in R_2 . By property 1), the last inequality is valid everywhere in R_2 .

Therefore, using Theorem 33 of ⁽¹⁾, for the planar variation W of the function $r_h(z)$ we have:

$$W(r_h, I) = \int_{-\infty}^{\infty} v(E_t^h) dt = \iint_I |\text{grad } r_h(x, y)| dx dy \leq S(I),$$

where $S(I)$ is the area of the fixed square $I \subset R_2$; $v(E_t^h)$ is the linear Hausdorff measure (⁽⁴⁾, p. 86) of the t -level E_t^h of the function $r_h(z)$. Further, on every square $I \subset R_2$ the planar variation of the function $r(z)$ is bounded, since it is a lower semicontinuous functional with respect to uniform convergence (⁽¹⁾, p. 90), and

$$W(r, I) \leq \lim_{h \rightarrow 0} W(r_h, I) \leq S(I).$$

It follows that the function $r(z)$ is absolutely continuous (in the sense of Krodnod). Indeed, let $\varepsilon > 0$ be arbitrary and let N be a bounded Borel set with measure less than $\delta = \varepsilon$. Cover N by a countable system of nonoverlapping squares $\{I_i\}$, $i = 1, 2, \dots$, with sum of areas $< \delta$. Then, by the complete additivity of planar variation on the class of Borel sets,

$$W(r, N) \leq W\left(r, \bigcup_i I_i\right) \leq \sum_i W(r, I_i) \leq \sum_i S(I_i) < \varepsilon.$$

By Theorem 32 of ⁽¹⁾, for a square $I \supset \bar{G}$ (\bar{G} is the closure of G) we have:

$$W(r, I) = \int_0^T v(E_t) dt = \iint_I |\text{grad } r(z)| dx dy, \quad (3)$$

whence, by virtue of equality (2),

$$\text{mes}_2 G \leq \int_0^T v(E_t) dt \leq \text{mes}_2 \bar{G}. \quad (3')$$

Let $\tau \in (0, T]$, $G^\tau = \{z : r(z, R_2 \setminus G) > \tau\}$, and $r_\tau(z) = r(z, R_2 \setminus G^\tau)$. It is easy to see that the t -level E_t^τ of the function $r_\tau(z)$ coincides, for $t \geq \tau$, with the $(t + \tau)$ -level of the function $r(z)$.

Applying inequality (3') to the function $r_\tau(z)$, we have

$$\text{mes}_2 \bar{G}^\tau \geq \int_0^{T-\tau} v(E_t^\tau) dt = \int_\tau^T v(E_t) dt,$$

and, since $\bar{G}^\tau \subset G$, therefore

$$\text{mes}_2 G \geq \int_0^T v(E_t) dt,$$

which together with (3') gives

$$\text{mes}_2 G = \int_0^T v(E_t) dt. \quad (4)$$

Since $v(E_t)$ is finite for almost all t (see (3)), Theorem 10 of ⁽¹⁾ permits us to assert that, for almost all $t \in (0, T]$, the t -level of the function

$r(z)$ consists of point components or entirely regular components, with $\nu(E_t) = \sum_\alpha \nu(E_t^\alpha) < \infty$. For such t the component E_t^α is locally connected ((1), p. 51) and contains no branch points; therefore E_t^α is homeomorphic to a circle ((1), p. 62), and $\nu(E_t^\alpha) = l_t^\alpha$ ((4), p. 190), which completes the proof of the theorem.

Remark 1. If the boundary ∂G of the set G has nonzero plane Lebesgue measure, then from (3) and (4) it follows that $\text{grad } r(z) = 0$ almost everywhere on ∂G .

Remark 2. For the case where G is a simply connected domain, in note (5) the supposition was made that the level set E_t has finite length for all t . Theorem 1 justifies this supposition for almost all $t \in (0, T]$, even when G is an arbitrary bounded open set. We also note that in the case considered by us, as in (5), a metric characterization of all t -levels is possible; namely, assertions analogous to assertions I–IV of the theorem cited in (5) are valid.

Quite analogously to Theorem 1 one proves

Theorem 2. Let G be a bounded open set in Euclidean space R_n , $x \in R_n$ an arbitrary point, and $r(x) = r(x, R_n \setminus G)$, where r is the distance in R_n . Then the $(n-1)$ -dimensional Hausdorff measure $\nu(E_t)$ of the set $E_t = \{x : r(x) = t\}$, as a function of t , is summable on the interval $(0, T]$, $T = \max_{x \in G} r(x)$, and moreover the n -dimensional Lebesgue measure is

$$\text{mes}_n G = \int_0^T \nu(E_t) dt. \quad (5)$$

From formula (5) it follows that the t -level E_t has finite $(n-1)$ -dimensional Hausdorff measure for almost all $t \in (0, T]$; however, apparently, it can be proved that $\nu(E_t) < \infty$ for all t .

Theorem 2 admits the following generalization.

Theorem 3. Let $M \subset R_n$ be an arbitrary measurable set, and let $F \subset R_n$ be such a closed set that $\text{mes}_n(F \cap M) = 0$, and let E_t^F be the t -level of the function $r_F(x) = r(x, F)$, where r is the distance in R_n ($x \in R_n$).

Then the $(n-1)$ -dimensional Hausdorff measure $\nu(E_t^F \cap M)$, as a function of t , is summable on the interval $(0, \infty)$, and moreover

$$\text{mes}_n M = \int_0^\infty \nu(E_t^F \cap M) dt. \quad (6)$$

It is evident that formula (6) generalizes the known formula relating the measure of a set in R_n to the measures of its sections (see (3), p. 371, or (6), p. 143). In the case $n = 2$, for almost all t , the Hausdorff measure $\nu(E_t^F \cap M)$ can be replaced by the Lebesgue measure $m_t(E_t^F \cap M)$ on the curve E_t^F .

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Note: Figure translations are in progress. See original paper for figures.

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