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Abstract

Full Text

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On Certain Classes of Regular Functions of Several Complex Variables

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The author, in (1, 4), studied, in the case of several complex variables, regular functions whose nature is similar to that of starlike univalent and convex univalent functions of one complex variable in the disk.

In the present note, also in the case of several complex variables, we carry out an investigation of regular functions whose nature is similar to that of p-valent, p-valently starlike with respect to the origin, and p-valently convex functions of one variable in the disk. In fact, all the investigations are carried out in the case of two complex variables, since in the case of n complex variables they are carried out in a completely analogous way.

1. Let D be a bounded complete bicircular domain with center at the point $(0,0)$, and let k_0 and k_0' be fixed finite numbers from the whole set of complex numbers. Suppose that in the domain D the function $F(w,z)$, $F(0,0)=1$, is regular. Consider the set $D \cap \{z = k_0 w\}$, which is the section of the domain D by the plane $z = k_0 w$.

Definition 1. We shall say that in the section $D \cap \{z = k_0 w\}$ the function $w^{pF}(w,z)$, where p is a natural number, is p-valent, p-valently starlike with respect to the origin (p-valently convex), if the function $W = w^{pF}(w, k_0 w)$, as a function of one complex variable w, is respectively p-valent in the corresponding disk (^{3}), p-valently starlike in this disk and maps it onto a domain starlike with respect to the point $W = 0$ (onto a convex domain).

Definition 1′. We shall say that in the section $D \cap \{w = k_0' z\}$ the function $z^{pF}(w,z)$ is p-valent, p-valently starlike with respect to the origin (p-valently convex), if the function $Z = z^{pF}(k_0' z, z)$, as a function of one complex variable z, is respectively p-valent in the corresponding disk (^{3}), p-valently starlike in this disk and maps it onto a domain starlike with respect to the point $Z = 0$ (onto a convex domain).

Remark 1. In Definition 1 (1′), by the corresponding disk (^{3}) is meant the projection of the section $D \cap \{z = k_0 w\}$ ($D \cap \{w = k_0' z\}$) onto the plane $z = 0$ ($w = 0$); if $k_0 = 0$ ($k_0' = 0$), then this will be the disk cut out by the domain D from the plane $z = 0$ ($w = 0$).

We also note that in Definitions 1 and 1′, by a convex domain is meant a domain such that, when its boundary is traversed in a definite direction, the tangent to it rotates in one direction.

Definition 2. We denote by $Q_D(p)$, $M_D(p)$ $(N_D(p))$ the classes of functions $F(w,z)$, $F(0,0)=1$, regular in the domain D, possessing the following properties: 1) in the section of the domain D by each plane from all possible analytic planes $z=kw$* the function $w^{pF}(w,z)$ is respectively p-valent, p-valently starlike with respect to the origin (p-valently convex); 2) in the section $D\cap\{w=0\}$ the function $z^{pF}(0,z)$ is respectively p-valent, p-valently starlike with respect to the origin (p-valently convex).

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* k runs through the entire set of complex numbers except ∞.

It follows from Definition 2 that $M_D^{(p)}\subset Q_D^{(p)}$, $N_D^{(p)}\subset Q_D^{(p)}$. For $p=1$ the introduced classes will be denoted respectively by Q_D, M_D (N_D).

2. Theorem 1. *In order that a function $F(w,z)$, $F(0,0)=1$, regular in the domain D, belong to the class $M_D^{(p)}$, respectively $N_D^{(p)}$, it is necessary and sufficient that in D*

$$\operatorname{Re}\left(\frac{L_p[F(w,z)]}{F(w,z)}\right)>0,\quad \text{respectively}\quad \operatorname{Re}\left(\frac{L_p\{L_p[F(w,z)]\}}{L_p[F(w,z)]}\right)>0,$$

where

$$L_p[F(w,z)]\equiv pF(w,z)+wF'_w(w,z)+zF'_z(w,z).$$

Theorem 2. *If a function $F(w,z)$, regular in the domain D, $F(0,0)=1$, satisfies in D the condition $|L_1[F(w,z)]-1|<1$, then $F(w,z)\in Q_D$.*

Theorem 1 is proved analogously to the same theorem of the author ([3]) in the case $p=1$, and Theorem 2–as the sufficiency part in the proof of the author' s theorem just mentioned ([3]), but with the use of a known proposition ([5]), $Satz 24$ for one variable.

Remark 2. The operators $L_1[F(w,z)]$, $L_p[F(w,z)]$ were introduced by A. A. Temlyakov ([6,7]).

3. We turn to the establishment of some estimates in the class Q_D. We first note that, as is known, every function $F(w,z)$, regular in the domain D, besides its representation in D by the series

$$F(w,z)=\sum_{m,n=0}^{\infty}a_{mn}w^m z^n,$$

can be represented in this domain by the diagonal series

$$F(w,z)=\sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}a_{k-l,l}w^{k-l}z^{l}\right).$$

Theorem 3. *If the function*

$$F(w,z)=\sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}a_{k-l,l}w^{k-l}z^{l}\right)\in Q_D,$$

then for $k=1,2,3$

$$A_k(D)\le(k+1)^2,\qquad B_k(D)\le k+1,$$

where

$$A_k(D)\equiv\sup_{(w,z)\in D}\sum_{l=0}^{k}|a_{k-l,l}|^2\,|w|^{2(k-l)}|z|^{2l},$$

$$B_k(D)\equiv\sup_{(w,z)\in D}\left|\sum_{l=0}^{k}a_{k-l,l}w^{k-l}z^{l}\right|,$$

and for $k>3$

$$A_k(D)<(2^{-1}e(k+1)+1.51)^2,\qquad B_k(D)<2^{-1}e(k+1)+1.51.$$

Proof. Let (w_0,z_0) be an arbitrary point of the domain D. Since the domain D is complete, for $|\xi|<1$ and $0\le t\le 2\pi$ the points $(\xi w_0,\xi z_0e^{-it})\in D$. One of two cases is possible: either $w_0\ne 0$, or $w_0=0$. Let $w_0\ne 0$. Then, proceeding from property 1) of the class Q_D and Definition 1, we establish that the function $\xi F(\xi w_0,\xi z_0e^{-it})=\xi+\cdots$, regular in the disk $|\xi|<1$, is univalent in $|\xi|<1$ for each fixed t from the segment $0\le t\le 2\pi$. Consequently ([8]), for $0\le t\le 2\pi$ we have:

$$\left|\sum_{l=0}^{k}a_{k-l,l}w_0^{k-l}z_0^{l}e^{-ilt}\right|\le k+1,\qquad k=1,2,3,\tag{1}$$

$$\left|\sum_{l=0}^{k}a_{k-l,l}w_0^{k-l}z_0^{l}e^{-ilt}\right|<2^{-1}e(k+1)+1.51,\qquad k>3,\tag{2}$$

moreover, from property 2) of the class Q_D and Definition 1′ it follows that inequalities (1), (2) are also preserved in the case $w_0=0$. Now, squaring both sides of each of the inequalities (1), (2) and then integrating each from the obtained inequalities in t over the interval from 0 to 2π, we easily arrive at the desired estimates for $A_k(D)$. Finally, putting $t = 0$ in (1), (2), we obtain inequalities from which the desired estimates for $B_k(D)$ follow.

Corollary 1. If the function

$$F(w,z) = \sum_{m,n=0}^{\infty} a_{mn} w^m z^n \in Q_D,$$

then

$$|a_{mn}| \leq (m+n+1)/d_{mn}(D), \qquad m+n = 1,2,3;$$

$$|a_{mn}| < [2^{-1}e(m+n+1) + 1.51]/d_{mn}(D), \qquad m+n > 3,$$

where

$$d_{mn}(D) = \sup_{(w,z)\in D} (|w|^m |z|^n).$$

Remark 3. If the domain D is the bicylinder $E\{|w| < R_1,\ |z| < R_2\}$, then, obviously,

$$\sup_{(w,z)\in D} \sum_{l=0}^{k} |a_{k-l,l}|^2 |w|^{2(k-l)} |z|^{2l} = \sum_{l=0}^{k} |a_{k-l,l}|^2 R_1^{2(k-l)} R_2^{2l}.$$

Consider the domain $\overline{D}_r = r\overline{D}$, where r is a positive number less than one.

Theorem 4. If the function $F(w,z) \in Q_D$, then in $\overline{D}_r$ we have the estimates:

$$(1+r)^{-2} \leq |F(w,z)| \leq (1-r)^{-2}; \tag{3}$$

$$(1+r)^{-3}(1-r) \leq |L_1[F(w,z)]| \leq (1-r)^{-3}(1+r). \tag{4}$$

We briefly outline the proof of the theorem. Let (w_0, z_0) be an arbitrary point of the domain $\overline{D}_r$. Take any number ρ satisfying the condition $r < \rho < 1$, and consider the domain $D_\rho = \rho D$. The point $(w_0, z_0) \in D_\rho$, and hence the point $(\rho^{-1}w_0, \rho^{-1}z_0) \in D$. It follows, by virtue of the fact that the domain D is complete, that for $|\zeta| < 1$ the points $(\rho^{-1}\zeta w_0, \rho^{-1}\zeta z_0) \in D$. One of two cases is possible: either $w_0 \neq 0$, or $w_0 = 0$. Let $w_0 \neq 0$. Then, proceeding from property 1) of the class Q_D and Definition 1, we establish that the function regular in the disk $|\zeta| < 1$,

$$\zeta F(\rho^{-1}\zeta w_0, \rho^{-1}\zeta z_0) = \zeta + \cdots,$$

is univalent in it. If $w_0 = 0$, then from property 2) of the class Q_D it follows that the function regular in the disk $|\zeta| < 1$,

$$\zeta F(0, \rho^{-1}\zeta z_0) = \zeta + \cdots,$$

is univalent in it. Finally, applying to these functions the corresponding estimates for univalent functions in the disk, we easily arrive at the desired estimates (3), (4).

Remark 4. Similarly to the results of this section, but using the corresponding results for p-valent functions in the disk, results of the same character can also be obtained for functions of the class $Q_D^{(p)}$.

4. We now turn to estimates in the classes $M_D^{(p)}$ and $N_D^{(p)}$.

Theorem 5. If the function

$$F(w, z) = \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} a_{k-l,l} w^{k-l} z^l \right) \in M_D^{(p)},$$

then for $k > 0$

$$A_k(D) \leq [(2p + k - 1)!/(2p - 1)!k!]^2,$$

$$B_k(D) \leq (2p + k - 1)!/(2p - 1)!k!.$$

The proof is analogous to the proof of Theorem 3, but with the use of estimates for the coefficients of p-valently starlike regular functions of one variable in the disk ([9]).

Corollary 2. If the function

$$F(w, z) = \sum_{m,n=0}^{\infty} a_{mn} w^m z^n \in M_D^{(p)},$$

then

$$|a_{mn}| \leq (2p + m + n - 1)!/(2p - 1)!(m + n)!d_{mn}(D), \qquad m + n > 0.$$

From Theorem 1 it follows that if the function $F(w, z) \in N_D^{(p)}$, then the function

$$\Phi(w, z) = p^{-1} L_p[F(w, z)] \in M_D^{(p)}.$$

Hence, on the basis of Theorem 5, we obtain Theorem 6:

Theorem 6. If the function

$$F(w, z) = \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} a_{k-l,l} w^{k-l} z^l \right) \in N_D^{(p)},$$

then for $k > 0$

$$A_k(D) \leqslant \left(\frac{(2p+k-1)!\, p}{(2p-1)!\, k!\, (k+p)} \right)^2, \qquad B_k(D) \leqslant \frac{(2p+k-1)!\, p}{(2p-1)!\, k!\, (k+p)}.$$

Corollary 3. If the function

$$F(w, z) = \sum_{m,n=0}^{\infty} a_{mn} w^m z^n \in N_D^{(p)},$$

then

$$|a_{mn}| \leqslant (2p+m+n-1)!\, p \,/\, (2p-1)!\, (m+n)!\, (m+n+p) d_{mn}(D), \qquad m+n > 0.$$

Relying on the results of Theorem 4, we establish the validity of Theorem 7:

Theorem 7. If the function $F(w, z) \in M_D^{(p)}$, then in $\overline{D}_r$ we have the estimates:

$$(1+r)^{-2p} \leqslant |F(w, z)| \leqslant (1-r)^{-2p},$$

$$p(1+r)^{-2p-1}(1-r) \leqslant |L_p[F(w, z)]| \leqslant p(1-r)^{-2p-1}(1+r).$$

A theorem of the same character also holds in the class $N_D^{(p)}$.

5. On the basis of Theorems 4 and 7, in the case of the domain D_0 (for the definition of the domain D_0 see (10)), which belongs to a sufficiently broad class of domains D, one establishes, respectively, propositions giving at each point $(w, z) \in D_0$ estimates analogous in form to the estimates of Theorems 4 and 7, but with r replaced in them by $\omega(|w|, |z|)$ (for the expression for $\omega(|w|, |z|)$ see (10)); moreover, in the case of the domains $C_\alpha \{(a|w|)^{\alpha^{-1}} + (b|z|)^{\alpha^{-1}} < 1\}$ $(a, b, \alpha > 0$ and $\alpha \leqslant 1)$

$$\omega(|w|,|z|) = \left[(a|w|)^{\alpha^{-1}} + (b|z|)^{\alpha^{-1}}\right]^{\alpha}. \tag{10}$$

Relying on these propositions in the case of the domains C_α, the same propositions are established in the case of the bicylinder E, where

$$\omega(|w|,|z|) = \begin{cases} |z|R_2^{-1}, & \text{for } (w,z) \in [(|w|R_1^{-1} \le |z|R_2^{-1}) \cap E], \\ |w|R_1^{-1}, & \text{for } (w,z) \in [(|w|R_1^{-1} > |z|R_2^{-1}) \cap E]. \end{cases}$$

All estimates in these propositions in the case of the domains C_1 are sharp, since there exist functions for which they can be attained. In the case of the domains C_α ($\alpha \ne 1$) and E, these estimates are sharp, respectively, on the sets $\{a|w| = b|z|\} \cap C_\alpha$ and $\{|w|R_1^{-1} = |z|R_2^{-1}\} \cap E$, for there also exist functions for which they can be attained.

Finally, we note that in the case of two complex variables A. A. Temlyakov [11] and, in the case of n complex variables, the author [12, 4], respectively, for relatively broad special classes of complete bicircular and multicircular domains, effectively computed

$$\sup_{(z_1,\ldots,z_n)\in D} \left(|z_1|^{k_1} \cdots |z_n|^{k_n}\right),$$

which makes the estimates in Corollaries 1, 2, 3 and analogous estimates in the case of n complex variables (for these classes of domains) effective.

6. In exactly the same way as in the case of the domain D, Definitions 1 ($1'$) and 2 are introduced in the case of a bounded complete circular domain K with center at the point $(0,0)$; in this case Theorems 1, 2, 4, 7 and their proofs for this case are completely preserved. Moreover, in Theorems 3, 5, 6 the estimates for the quantity $B_k(D)$, with D replaced by K, remain valid also under the conditions of these theorems in the case of the domain K.

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Note: Figure translations are in progress. See original paper for figures.