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Abstract

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THE INITIAL-DATA PROBLEM FOR THE KINETIC EQUATION IN A HOMOGENEOUSLY EXPANDING (COMPRESSING) PLASMA

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In the works (^{1,2}) A. A. Nikol' skii found solutions of the Boltzmann kinetic equation for various cases of intermolecular interaction, which reduce the initial-data problem for a homogeneously expanding or homogeneously compressing gas to the corresponding Cauchy problem in a homogeneous medium at rest. Below a similar solution will be obtained for the system of kinetic equations of a plasma with the Landau collision integral.

As is known, the equations of gas dynamics for a gas occupying unbounded space, in the absence of force fields, admit a motion in which the velocities of the medium are distributed according to the law

$$\mathbf{v}_0 = \mathbf{r}/t, \tag{1}$$

and the density and temperature depend only on time,

$$n = n_0 t^{-3}, \qquad T = T_0 t^{-2}. \tag{2}$$

(We have introduced dimensionless time and take the initial values n_0, T_0 to be specified at $t = 1$.)

Substitution of formulas (1), (2) into the equations of plasma dynamics (see (^{3}), p. 191) shows that, under the assumptions made above, such a motion—homogeneous expansion—is possible also for a plasma.

The system of kinetic equations for electrons ($\alpha = e$) and singly charged ions ($\alpha = i$) is written in the form (^{4,5})

$$\frac{\partial f^\alpha}{\partial t} + v_{\alpha l}\frac{\partial f^\alpha}{\partial x_l} = -\sum_\beta \frac{\partial}{\partial v_{\alpha k}} j_k^{\alpha/\beta}, \tag{3}$$

where

$$j_k^{\alpha/\beta} = 2\pi\lambda \frac{e^4}{m_\alpha} \int U_{lk} \left(\frac{f^\alpha}{m_\beta} \frac{\partial f^\beta}{\partial v_{\beta k}} - \frac{f^\beta}{m_\alpha} \frac{\partial f^\alpha}{\partial v_{\alpha k}} \right) d\mathbf{v}_\beta; \tag{4}$$

$$U_{lk} = \frac{\delta_{lk}}{v} - \frac{v_l v_k}{v^3}, \qquad v_l = v_{\alpha l} - v_{\beta l}; \tag{5}$$

$$\lambda = \ln \frac{D}{\rho_0} = \ln \frac{(T/8\pi n e^2)^{1/2}}{(e^2/3T)}. \tag{6}$$

In what follows we omit the indices (α, β) indicating the species of particles. For equation (3) we consider the following initial problem in a homogeneously expanding gas: at the instant $t = 1$ a spatially homogeneous distribution $f(t, \mathbf{r}, \mathbf{v}) = \varphi(\mathbf{u})$ is specified, where $\mathbf{u} = \mathbf{v} - \mathbf{r}/t$ is the peculiar velocity of the particle.

Let us pass to the new independent variables $(t, \mathbf{r}, \mathbf{u})$ and use the independence of the distribution function, expressed in these variables,

from the coordinates. In the new variables the collision integral will not change, and the left-hand side of equation (3) is transformed into

$$\frac{\partial f}{\partial t} - \frac{1}{t} u_k \frac{\partial f}{\partial u_k}. \tag{7}$$

Let us then pass to the new variables $\tau(t)$, $\mathbf{w} = \mathbf{u}t$. Denote $f(t, \mathbf{u}) = F(\tau, \mathbf{w})$. Expression (7) is transformed in the same way as in $(1, 2)$. In order to write the collision integral in the new variables, we make a change of variables in formulas (4)–(5), and in (6) pass to the time-independent initial values n_0, T_0 by formulas (2).

We obtain

$$U'_{lk} = tU^0_{lk}, \qquad \lambda = \lambda^0 - \frac{3}{2} \ln t, \qquad j_l = \left(1 - \frac{3}{2\lambda^0} \ln t\right) t^{-1} j^0_l. \tag{8}$$

Here all quantities with the superscript zero are expressed in terms of the vector $\mathbf{w}$, $F(\tau, \mathbf{w})$, and the constants n_0, T_0, just as the corresponding quantities without the superscript are expressed in terms of $\mathbf{v}, f, n, T$, and do not depend on time.

Equation (3) is transformed into

$$\frac{d\tau}{dt} \frac{\partial F}{\partial \tau} = -\left(1 - \frac{3}{2\lambda^0} \ln t\right) \frac{\partial}{\partial w_l} j^0_l. \tag{9}$$

Let us subject $\tau(t)$ to the conditions

$$\frac{d\tau}{dt} = 1 - \frac{3}{2\lambda^0}\ln t, \qquad \tau(1) = 0. \tag{10}$$

Equation (9) is simplified and is reduced to the equation in a stationary homogeneous gas with the same initial condition, but at $\tau = 0$; τ plays the role of time:

$$\frac{\partial F}{\partial \tau} = -\frac{\partial}{\partial w_l} j_l^0, \qquad F(0, \mathbf{w}) = \varphi(\mathbf{w}) \equiv \varphi(\mathbf{u}). \tag{11}$$

From equation (11) and the transformations carried out it follows that the solution of the initial-value problem for a uniformly expanding gas can be written in the form

$$f(t, \mathbf{r}, \mathbf{v}) = F(\tau, \mathbf{w}), \tag{12}$$

where $\mathbf{w} = \mathbf{v}t - \mathbf{r}$,

$$\tau = (t-1)\left(1 - \frac{3}{2\lambda^0}\right) - \frac{3}{2\lambda^0} t \ln t$$

(τ is the integral of equation (10)).

For the analysis of solution (12) we use Boltzmann' s H-theorem in the form asserting the absence of negative sources of entropy (6)

$$\partial H/\partial t + \operatorname{div} \mathbf{S} \geq 0. \tag{13}$$

Here

$$H[f] = -\int f \ln f \, d\mathbf{v} \quad \text{(entropy)},$$

$$\mathbf{S}[f] = -\int \mathbf{v} f \ln f \, d\mathbf{v} \quad \text{(entropy flux)}.$$

For the kinetic equations of a plasma with a collision integral of more general form (sometimes called the Balescu collision integral), the H-theorem was proved in works (7,8). The Landau collision integral can be obtained from the Balescu integral by artificially restricting the domain of integration on the side of large (greater than the Debye radius D) and small (less than the "distance of strong interaction" ρ_0) impact parameters; moreover, the magnitude of the discarded terms is

$1/\lambda$ of the magnitude of the integral. Both of the collision integrals mentioned above were obtained under the assumption of a homogeneous state of the plasma with constant values of density and temperature, but the proof of the H-theorem for the Balescu integral does not require such an assumption. This proof can be carried over to the Landau collision integral if it is assumed that the Coulomb logarithm remains positive throughout the motion, $0 < \lambda$.

Expression (13) for the motion under consideration, upon passing to the proper velocity, is transformed into the form

$$\frac{\partial H^*}{\partial t} + \operatorname{div} \mathbf{S}^* + \frac{3}{t} H^* \geqslant 0, \tag{14}$$

where

$$H^*[f] = -\int f \ln f \, d\mathbf{u} = H[f],$$

$$\mathbf{S}^*[f] = -\int \mathbf{u} f \ln f \, d\mathbf{u} = \mathbf{S}[f] - \frac{\mathbf{r}}{t} H[f].$$

In a motionless homogeneous gas $0 < \lambda^0$, and the entropy principle is written in the form:

$$\partial H[F]/\partial \tau \geqslant 0, \qquad \operatorname{div} \mathbf{S}[F] = 0. \tag{15}$$

With the aid of solution (12) we obtain the transformation formulas

$$H^*[f] = t^{-3} H[F], \qquad \mathbf{S}^*[f] = t^{-4} \mathbf{S}[F]. \tag{16}$$

Substitution of (16) into (14) gives

$$\frac{1}{t^3} \frac{\partial H[F]}{\partial \tau} \frac{d\tau}{dt} \geqslant 0.$$

This inequality does not contradict inequality (15) if $d\tau/dt > 0$. The latter condition is violated beginning with $t = \exp\{2/\lambda^0\}$, where the function τ attains a maximum. However, solution (12) is exact, and its validity for all values of t can be verified by direct substitution into equation (3).

The reason that for $t > t_1$ the entropy principle is not satisfied lies in the approximate character of the kinetic equations being solved. As was said, the Landau integral is obtained with accuracy $1/\lambda$ (some authors dispute this estimate, for example ([9])). Under homogeneous expansion the value of the Coulomb logarithm, according to (8), decreases, and at the same time the "accuracy" of the collision integral decreases (if one assumes that the Landau integral is applicable in the case of nonconstant n and T). Already at $t = \exp\{2/3\lambda^0 - 1\}$ the Coulomb logarithm $\lambda = 1$, and the "accuracy" provided by the Landau integral is insufficient.

Finally, for $t > t_1$ the Coulomb logarithm becomes negative, the "strong-interaction distance" ρ_0 exceeds the Debye radius, and the Landau integral loses its physical meaning. It is in this region that violation of the principle of increasing entropy occurs. It should be noted that the solutions of the initial-value problem for the Boltzmann equation, contained in ([1],[2]), as verification shows, do not contradict the entropy principle for all $1 \leqslant t < \infty$ under any law of interaction of molecules.

We give, without derivation, the solution of the analogous problem for homogeneous compression of a plasma. Here the initial conditions should be regarded as prescribed at $t = -1$, with the interval of values of t being $[-1, 0)$. The mean velocities of the particles are distributed according to (1), while the density and temperature vary according to the law:

$$n = n_0 |t|^{-3}, \qquad T = T_0 t^{-2}.$$

The solution is expressed in the form

$$f(t, \mathbf{r}, \mathbf{v}) = F(\tau, \mathbf{w}),$$

where $\mathbf{w} = \mathbf{u}|t|$,

$$\tau = (1 - |t|)\left(1 + \frac{3}{2\lambda^0}\right) + \frac{3}{2\lambda^0}|t| \ln |t|.$$

On the entire interval $[-1, 0)$ the function τ increases and the entropy principle is satisfied. As $t \to 0$, $\tau \to \tau_k = 1 + 3/2\lambda^0$. Thus, under limiting compression, $f(0, \mathbf{r}, \mathbf{v}) = F(\tau_k, \mathbf{w})$, and the equilibrium state $F(\infty, \mathbf{w})$ is not reached.

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CITED LITERATURE

1. A. A. Nikol' skii, *Dokl. Akad. Nauk SSSR* **151**, No. 2 (1963).

2. A. A. Nikol' skii, *Dokl. Akad. Nauk SSSR* **151**, No. 3 (1963).
3. S. I. Braginskii, *Problems of Plasma Theory* **1**, 1963, p. 183.
4. B. A. Trubnikov, ibid., p. 98.
5. L. D. Landau, *ZhETF* **7**, issue 2 (1937).
6. A. Sommerfeld, *Thermodynamics and Statistical Physics*, IL, 1955.
7. Yu. L. Klimontovich, *Statistical Theory of Nonequilibrium Processes in Plasma*, Moscow, 1964.
8. I. Prigogine, *Nonequilibrium Statistical Mechanics*, IL, 1964.
9. T. Koga, *Phys. Fluids* **5**, No. 6 (1962).

Note: Figure translations are in progress. See original paper for figures.