ON THE BASIC SPATIAL BOUNDARY-VALUE PROBLEMS FOR COMPOSITE ISOTROPIC ELASTIC MEDIA*

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Abstract

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ON THE BASIC SPATIAL BOUNDARY-VALUE PROBLEMS FOR COMPOSITE ISOTROPIC ELASTIC MEDIA*

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1. Let D be a three-dimensional domain bounded by a surface S_0; let D_i $(i = 1, \dots, n)$ be a domain lying inside D, together with its closure $\overline{D}_i$, and bounded by a surface S_i. Let $\overline{D}_i \cap \overline{D}_j = \emptyset$ $(i, j = 1, \dots, n;\ i \neq j)$,

$$D_0 = D \setminus \bigcup_{i=1}^{n} \overline{D}_i;$$

D_{n+1} is the complement of D to the whole space.

Suppose that the domain D_i $(i = 0, \dots, n)$ is filled with an isotropic homogeneous elastic medium characterized by the constant Lamé parameters λ_i and μ_i. Introduce the notation:

$$\Delta_i u = \mu_i \sum_{j=1}^{3} \frac{\partial^2 u}{\partial x_j^2} + (\lambda_i + \mu_i)\, \mathrm{grad}\, \mathrm{div}\, u,$$

$$T_i u = 2\mu_i \frac{\partial u}{\partial \nu} + \lambda_i \nu\, \mathrm{div}\, u + \mu_i [\nu\, \mathrm{rot}\, u],$$

where $u = (u_1(x), u_2(x), u_3(x))$ is a vector; $\nu = (\nu_j)$ is the unit normal exterior with respect to D_i at the point x, for $x \in S_i$ $(i = 0, \dots, n)$; $x = (x_i)$ is a point of three-dimensional Euclidean space; x_1, x_2, x_3 are its Cartesian coordinates.

The basic boundary-value problems of the theory of elasticity for the indicated composite medium are posed as follows:

Find a vector $u(x)$, continuous together with its first derivatives with respect to the Cartesian coordinates of the point x in the closed domains $\overline{D}_i$ $(i = 0, \dots, n)$, possessing continuous second derivatives in the domains D_i $(i = 0, \dots, n)$, and satisfying the conditions:

$$1) \quad \Delta_i u(x) = \psi_i(x) \quad \text{for } x \in D_i\ (i = 0, \dots, n);$$

$$2') \quad u^+(x) = f_0(x)$$

or

$$2'') \quad (T_0 u)^+(x) = f_0(x) \quad \text{for } x \in S_0;$$

$$3) \quad u^+(x) - u^-(x) = f_i(x) \quad \text{and} \quad (T_i u)^+(x) - (T_0 u)^-(x) = \varphi_i(x) \quad \text{for}$$

$$x \in S_i \ (i = 1, \dots, n).$$

The symbols $u^+(x)$ for $x \in S_i$ $(i = 0, \dots, n)$ and $u^-(x)$ for $x \in S_i$ $(i = 1, \dots, n)$ denote the limiting values of u at the point x from D_i and D_0, respectively. $(T_i u)^+(x)$ and $(T_i u)^-(x)$ for $x \in S_i$ $(i = 0, \dots, n)$ denote the limiting values of the expressions $T_i u$ and $T_0 u$ at the point x from D_i and D_0, respectively. ψ_i $(i = 0, \dots, n)$ are given vectors on D_i; f_i $(i = 0, \dots, n)$ and φ_i $(i = 1, \dots, n)$ are given vectors on S_i.

In the case when conditions 1), $2'$), 3) are fulfilled, we have the first basic problem, and in the case of 1), $2''$), 3), the second.

If now the domain D_{n+1} is occupied by one elastic medium, D_0 by another medium, and D_i $(i = 1, \dots, n)$ are voids, then the vector u must satisfy condition 1) for $i = 0$ and $i = n + 1$, and condition 3) for $i = 0$. Conditions $2'$) and $2''$) are replaced in this case, respectively, by the conditions

$$2^* \quad u^-(x) = f_i(x);$$

$$2^{**} \quad (T_0 u)^-(x) = f_i(x) \quad \text{for } x \in S_i \ (i = 1, \dots, n).$$

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Moreover, the vector u must be regular at infinity. If the domain D coincides with the whole space, then conditions $2'$) and $2''$) are replaced by the condition of regularity of the vector u at infinity.

2. All the problems presented above were studied by V. D. Kupradze ([1,2]) under the assumption that the Poisson coefficients of the media considered are close. In this note another method is proposed for studying these problems. In this method no restrictions are imposed on the Poisson coefficients.

3. Of the problems under consideration, the second boundary-value problem presents the greatest difficulty for investigation. We shall study it in detail. Suppose, for simplicity, that $\psi_i = 0$, $n = 1$, S_0 is a surface of class ([3]) $\mathcal{L}(1,\alpha)$, and S_1 is of class $\mathcal{L}(2,\alpha)$, $f_0 \in H(\beta)$ on S_0, $\varphi_1 \in H(\beta)$ on S_1, $f_1 \in H(1,\beta)$ on S_1 (see ([3])). α and β are positive constants, with $\alpha > \beta$. We choose the origin of coordinates in the domain D_0. All these conditions will henceforth be assumed satisfied.

We seek the solution of the second problem in the form

$$u(x) = \int_{S_0} \Gamma_0(x,y) g_1(y)\, d_{yS} +$$

$$+ \int_{S_1} \left[\Gamma_0(x,y) B_0 g_3(y) - P_0(x,y) A_0 g_2(y)\right] d_{yS} \quad \text{for } x \in D_0; \tag{1}$$

$$u(x) = \int_{S_1} \left[\Gamma_1(x,y) B_1 g_3(y) - P_1(x,y) A_1 g_2(y)\right] d_{yS} \quad \text{for } x \in D_1, \tag{2}$$

where

$$A_k = \mu_1\mu_0[\mu_k(\mu_1+\mu_0)]^{-1}; \quad B_k = a_1 a_0 [a_k(a_1+a_0)]^{-1}; \quad a_k = \mu_k(\lambda_k + 2\mu_k)^{-1};$$

$$\Gamma_k = (4\pi\mu_k r)^{-1} \left\|2\delta_{ij} - b_k r_i r_j\right\|; \quad b_k = (\lambda_k+\mu_k)(\lambda_k+2\mu_k)^{-1};$$

$$P_k = (2\pi)^{-1} \left\| \left[(1-c_k)\delta_{ij} + 3c_k r_i r_j\right] r_n + (a_1+a_0)^{-1}\left[(-1)^k(a_1-a_0) + 2a_1a_0\right](n_i r_j - n_j r_i) r^{-2} \right\|;$$

$$c_k = 2b_k a_1 a_0 [a_k(a_1+a_0)]^{-1}; \quad k = 0,1; \quad r^2 = \sum (x_i - y_i)^2; \quad y = (y_i);$$

δ_{ij} is the Kronecker symbol; $n = (n_i)$ is the normal at the point y exterior with respect to D_k for $y \in S_k$ (k

$$r_i = \frac{\partial r}{\partial x_i}; \qquad r_n = \frac{\partial}{\partial n}\frac{1}{r}; \qquad r_{ij} = \frac{\partial^2 r}{\partial x_i \partial x_j};$$

$g_i = (g_{ij})$ are the unknown vectors; g_1 and g_3 are sought in the class $H(\gamma)$ on S_0 and S_1, respectively, and g_3 is sought in the class $H(1,\gamma)$ on S_1; γ is some positive number.

To determine g_1, g_2, and g_3, the following system of integral equations is obtained:

$$g_i(x) = \sum_{j=1}^{3} \int_{S_{j-1}} k_{ij}(x,y) g_j(y)\, d_{yS} = \Phi_i(x); \tag{3}$$

$x \in S_0$ for $i = 1$ and $x \in S_1$ for $i = 2,3$. Here $S_2 = S_1$;

$$k_{11} = (2\pi)^{-1} \left\|(a_0\delta_{ij} + 3b_0 r_i r_j) r_\nu + a_0(\nu_i r_j - \nu_j r_i) r^{-2}\right\|,$$

$$k_{12} = \mu_0(2\pi)^{-1}\left\|\nu_i r_n - \nu_j r_{ni} - \delta_{ij} r_{n\nu} - \left\{[r_{ij}\Sigma n_e \nu_e + \nu_i r_{nj} - n_j r_{\nu i} + (n_j\nu_i - n_i\nu_j) r^{-3} - 3n_i r^{-1} r_j r_\nu] a_2 \right.\right.$$
$$\left.\left. +12 b_0 a_1 [\nu_j r_i r^{-1} - r(\delta_{ij} - 5r_i r_j) r_\nu] r_n - 3(3a_1 - a_0) n_j r_i r_\nu r^{-1}\right\} (a_0 + a_1)^{-1}\right\|,$$
$$k_{22} = [2\pi(\mu_0+\mu_1)(a_0+a_1)]^{-1}\left\| r_n \left[6(\mu_1 a_1 b_0 - \mu_0 a_0 b_1) r_i r_j + a_3\delta_{ij}\right]\right.$$
$$\left. + [(a_1 - a_0)(\mu_0+\mu_1) + 2a_1 a_0(\mu_1 - \mu_0)]\,(\nu_j r_i - \nu_i r_j)\right\|,$$
$$k_{23} = a_4(2\pi)^{-1}\left\|2(a_0\mu_0 - a_1\mu_1)\delta_{ij} r^{-1} - a_5 r_{ij}\right\|,$$
$$k_{32} = a_6\left\|(\nu_j r_n - n_j r_\nu) r_i r^{-1} - (\delta_{ij} - 5r_i r_j) r r_\nu r_n\right\|,$$
$$k_{33} = 3(a_0 - a_1)[2\pi(a_0+a_1)]^{-1}\left\| r_i r_j r_\nu\right\|;$$
$$a_3 = (\mu_0+\mu_1)(a_0+a_1) + 2a_1 a_0(\mu_1-\mu_0), \quad a_4 = [4\pi\mu_1\mu_0(a_0+a_1)]^{-1},$$
$$a_5 = a_0\mu_0 b_1 - a_1\mu_1 b_0, \quad a_6 = 6(a_1 - a_0)\mu_1\mu_0[\pi(\mu_0+\mu_1)(a_0+a_1)]^{-1},$$
$$k_{13} = k_{11}B_0, \qquad k_{31} = k_{11}, \qquad k_{21} = -\Gamma_0, \qquad \Phi_1 = f_0, \quad \Phi_2 = f_1, \quad \Phi_3 = \varphi_1,$$
$$r_{nj} = \frac{\partial^2}{\partial n\,\partial x_j}\frac{1}{r}, \qquad r_{\nu j} = \frac{\partial^2}{\partial \nu\,\partial x_j}\frac{1}{r}, \qquad r_\nu = \frac{\partial}{\partial\nu}\frac{1}{r}, \qquad r_{n\nu} = \frac{\partial^2}{\partial\nu\,\partial x}\frac{1}{r}.$$

Let $a_1(x) = 1$ for $x \in S_0$ and $a_1(x) = 0$ for $x \in S_1$; $a_2(x) = a_3(x) = 0$ for $x \in S_0$ and $a_2(x) = a_3(x) = 1$ for $x \in S_1$; $\Phi_1(x) = \Phi_1(x)$ for $x \in S_0$ and $\Phi_1(x) = 0$ for $x \in S_1$; $\Phi_2(x) = \Phi_3(x) = 0$ for $x \in S_0$ and $\Phi_2(x) = \Phi_2(x)$, $\Phi_3(x) = \Phi_3(x)$ for $x \in S_1$.

Consider the system of equations

$$h_i(x) + a_i(x)\sum_{j=1}^{3}\int_{S_0\cup S_1} a_j(y) k_{ij}(x,y) h_j(y)\, d_{yS} = \Phi_i(x), \tag{4}$$

where $h_i = (h_{ij})$ is the vector sought on $S_0 \cup S_1$ $(i, j = 1, 2, 3)$.

(4) is a system of two-dimensional singular integral equations distributed over $S_0 \cup S_1$. It is easy to establish the connection between the solutions of systems (3) and (4).

Just as in note ([4]), the symbolic matrix $\Phi(x, \vartheta)$ and its determinant are computed:

$$\det\Phi(x,\vartheta) = [1 - a_1(x)a_0^2][1 - a_2(x)\chi^2],$$

where

$$\chi = \frac{a_1 - a_0}{a_1 + a_0} + \frac{2a_1 a_0}{a_1 + a_0}\frac{\mu_1 - \mu_0}{\mu_1 + \mu_0}.$$

Let us study system (4) in the space $L_p(S_0 \cup S_1)$, where p is an arbitrary number greater than 1. From the inequality

$$1 - \chi^2 = 4a_1 a_0 (a_1 + a_0)^{-2} (\mu_1 + \mu_0)^{-2} [\mu_0^2 \nu_0 (1 + a_1) + \mu_1^2 \nu_1 (1 + a_0) +$$

$$+ 2\mu_1 \mu_0 (1 + a_1 a_0) > 0$$

it follows that $\det \Phi(x, \vartheta) > 0$. Consequently, system (4) is of normal type in the space $L_p(S_0 \cup S_1)$.

From the arguments given in notes ([3],[4]), the validity of the following propositions follows:

For system (4) the Fredholm theorems are valid in the space $L_p(S_0 \cup S_1)$.

Every solution h_1, h_2, h_3 of system (4) belonging to the class $L_p(S_0 \cup S_1)$ satisfies the condition $h_1 \in H(\beta)$, $h_2 \in H(1, \beta)$, $h_3 \in H(\beta)$.

Taking these propositions into account, as well as the boundary properties of the potentials of the theory of elasticity ([5]), by a method analogous to that applied in notes ([6–8]), one can make the following conclusion.

For solvability of system (3) it is necessary and sufficient that the conditions

$$\int_{S_0} f_0(y)\, d_{yS} - \int_{S_1} \varphi_1(y)\, d_{yS} = 0, \tag{5}$$

$$\int_{S_0} [R(y) f_0(y)]\, d_{yS} - \int_{S_1} [R(y) \varphi_1(y)]\, d_{yS} = 0, \tag{6}$$

$$R(y) = (y_1, y_3, y_3).$$

From the above reasoning the main proposition of this note follows easily:

Theorem. *The second basic problem for composite elastic media in the formulation given above is solvable only if conditions (5) and (6) are satisfied. When these conditions are satisfied, which express the vanishing of the principal vector and principal moment of the external forces acting on the boundary of the domain D_0, the solution of the problem is given, up to a rigid displacement, in the form (1) and (2), where g_1, g_2, and g_3 are a solution of system (3).*

4. In an analogous way, the fundamental spatial boundary-value problems for composite isotropic elastic media are studied in the case of steady-state oscillations, as well as inhomogeneous problems of diffraction of electromagnetic waves.

5. We note that the problems studied above are boundary-value problems for special elliptic systems with discontinuous coefficients. For a single general elliptic equation this problem has been studied by many authors (see, for example, the work [9], which also contains fairly detailed references on these questions). The method used in the present note is also applicable to a single equation with discontinuous coefficients. In this case one can study not only the Dirichlet and Neumann problems, but also the problem with an oblique derivative, provided that the directions along which the derivatives are prescribed do not lie in the tangent planes to the boundary.

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Note: Figure translations are in progress. See original paper for figures.