ON THE REPRESENTATION OF ANALYTIC PERIODIC FUNCTIONS AS A SUM OF SQUARES

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Abstract

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ON THE REPRESENTATION OF ANALYTIC PERIODIC FUNCTIONS AS A SUM OF SQUARES

In the theory of differential operators it is often convenient to represent a positive definite operator with constant coefficients in the form

$$L * \varphi = \sum_{\alpha} [L_\alpha(x) * L_\alpha^*(-x)] * \varphi. \tag{1}$$

Such a representation, for example, for the polyharmonic operator

$$(-\Delta)^m = \sum_{|\alpha|=m} D^\alpha(x) * D^\alpha(-x), \tag{2}$$

makes it possible to obtain Green' s formula

$$(-1)^m \int \varphi \Delta \psi \, dx = \int \sum_{|\alpha|=m} D^\alpha \varphi D^\alpha \psi \, dx \tag{3}$$

and gives the investigator an apparatus by means of which it is easy to construct various symmetric extensions of the polyharmonic operator. Representations of type (1) are analogues of the simplest representations of nonnegative symmetric matrices A by the formula

$$A = BB^*. \tag{4}$$

The same apparatus is also useful for extending difference operators.

The Fourier transform of generalized functions reduces the problem of representing the operator L in the form (1) to the dual problem for the functions

$$\widetilde{L}(p) = \int e^{2\pi i p x} L(x) \, dx. \tag{5}$$

In the case when the operator $L(x)$ is a difference operator with nodes at the points of a regular lattice, i.e. has the form

$$L(x) = \sum_{\gamma=1}^{N} l_\gamma \delta(x - hH\gamma),$$

the Fourier transform $\widetilde{L}(p)$ will be a periodic function with period matrix $h\bar{H}$, where $|H| = 1$.

Representation of the operator A by formula (1) with operators $L_\alpha(x)$ decaying exponentially at infinity is equivalent to the representation

$$\widetilde{L}(p) = \sum_{\alpha} |\widetilde{L}_\alpha(p)|^2, \tag{6}$$

where $L_\alpha(p)$ are real analytic periodic functions.

The purpose of the present note is to indicate some sufficient conditions for the existence of formula (6).

Let us introduce some notation. Let $\mathfrak{A}$ be the space of integer-valued vectors with nonnegative components with the natural partial ordering of its elements. A set $H \subset \mathfrak{A}$ is **complete from above** if, together with each element a, it contains all elements following it, and **complete from below** if, together with a, it contains all elements preceding it.

A complete-from-above set $\dot{H}$ contains only a finite number of extreme elements, i.e., elements that have no predecessors in it. It is representable in the form

$$H = \bigcup_j K^{(\alpha(j))}, \qquad j = 1, 2, \dots, Q, \tag{7}$$

where $K^{(\alpha(j))}$ is the set of all elements following $\alpha^{(j)}$.

Let ρ be a regular n-digit binary fraction and let $\bar{\rho} = 1 - 2^{-n} - \rho$. In the expressions ρ and $\bar{\rho}$, ones and zeros change places. The space $\mathfrak{A}$ is represented in the form

$$\mathfrak{A} = \mathfrak{A}^{(\rho)} \times \mathfrak{A}^{(\bar{\rho})}, \tag{8}$$

where $\mathfrak{A}^{(\rho)}$ is the space consisting of the coordinates whose numbers correspond to those digits of ρ in which ones stand.

Put $\gamma \in \mathfrak{A}^{(\bar{\rho})}$. By $\mathfrak{A}^{(\rho)}(\gamma)$ we denote the set of elements of $\mathfrak{A}$ for which the coordinates marked by ones in the expression $\bar{\rho}$ assume the prescribed values γ. The set $\bar{H}^{(\rho)}(\gamma) = H \cap \mathfrak{A}^{(\rho)}(\gamma)$ will be called a section of H, and the set $\mathfrak{A}^{(\rho)}(\gamma)$,

$$\bigcup_{\gamma} H^{(\rho)}(\gamma) = \overset{\circ}{H}^{(\rho)} \tag{9}$$

the projection of H onto $\mathfrak{A}^{(\rho)}$.

The class of functions $f(\lambda)$, defined in $E^{(n)}$, analytic and real for real λ, periodic with period 2π in each variable, will be denoted by π. To each function $A(\lambda) \in \pi$ we put in correspondence the set of coefficients of its expansion in the Maclaurin series

$$A(\lambda) = \sum_{\alpha} a(\alpha) \frac{\lambda^{\alpha}}{\alpha!}. \tag{10}$$

The function $a(\alpha)$ is defined on the set A. The set of those α for which $a(\alpha) \neq 0$ is completed to a minimal complete-from-above H, which we shall call the Maclaurin structure of the function φ. $H = Mc\{\varphi\}$. We shall consider traces of the function $a(\alpha)$ corresponding to $A(\lambda)$. These traces, for a given $\gamma \in \mathfrak{A}^{(\bar{\rho})}$, will be denoted $a^{(\rho)}(\gamma/\beta)$, where $\beta \in \mathfrak{A}^{(\rho)}$. To each trace there corresponds a certain trace of a derivative of $A(\lambda)$ on the space $E^{(\rho)}$ of lower dimension:

$$D^{\gamma} A(\nu/\mu) \leftrightarrow \mathfrak{A}^{(\rho)}(\gamma, \beta), \tag{11}$$

where $E^{(n)} = E^{(\rho)} \times E^{(\bar{\rho})}$, $\lambda = \nu/\mu$, $\nu \in E^{(\bar{\rho})}$, $\mu \in E^{(\rho)}$. The set of such traces is naturally partially ordered. A trace of higher dimension follows all traces of lower dimension contained in it.

A subset of traces of the form $\mathfrak{A}^{(\rho)}(\gamma^*, \beta)$, where γ^* are the extreme elements of the projection $\bar{H}^{(\rho)}$, will be called the skeleton of the function $a(\alpha)$ and, correspondingly, $D^{\gamma^*} A(0, \mu)$ the skeleton of the function A.

We shall say that a function $A(\lambda) \in \pi$ is positive in the main if it vanishes nowhere except at those points where at least one of the coordinates λ_i is a multiple of 2π.

Theorem 1. *If, for the function $A(\lambda)$, all extreme elements of the Maclaurin structure are even and the skeleton is positive in the main, then $A(\lambda)$ is representable in the form of a sum of squares of real periodic functions*

$$A(\lambda) = \sum_{\alpha} \left[B_{\alpha} \left(\frac{\lambda}{2} \right) \right]^2, \tag{12}$$

where all $B_{\alpha}(\lambda) \in \pi$.

Let us outline the proof. If $Mc\{A\}$ has one extreme element, then $A(\lambda)$ is representable in the form

$$A(\lambda) = \sin^{2\alpha_1}\frac{\lambda_1}{2}\cdots\sin^{2\alpha_n}\frac{\lambda_n}{2}\,B(\lambda), \tag{13}$$

where $B(\lambda)$ is everywhere positive and, consequently, its square root can be extracted, and the theorem is obvious.

In the case where there are several extreme elements, $\alpha^{(j)}$, $j = 1, 2, \dots, Q$, $A(\lambda)$ can always be represented as the sum of a finite number of such functions with one extreme element:

$$A(\lambda) = \sum_{\alpha} A_\alpha(\lambda). \tag{14}$$

Let us establish this. We divide the extreme elements arbitrarily into two groups $\beta^{(1)}, \beta^{(2)}, \dots, \beta^{(s)}$, $\gamma^{(1)}, \dots, \gamma^{(r)}$, and show that $A(\lambda)$ is representable in the form

$$A(\lambda) = A_1(\lambda) + A_2(\lambda), \tag{15}$$

where $A_1(\lambda)$ and $A_2(\lambda)$ satisfy all the conditions of the theorem, with A_1 having as extreme points of the Maclaurin structure $\beta^{(1)}, \dots, \beta^{(s)}$, and A_2—all the remaining ones.

The proof is based on three lemmas.

Lemma 1. For any s-dimensional vector $\gamma^* \in \mathfrak{A}$ there exists a function of s variables $A(\lambda) \subset \mathfrak{p}$, for which all traces of derivatives

$$D^\nu \varphi(0/\mu) \quad \text{for} \quad \gamma_j \leqq \gamma_j^* \tag{16}$$

assume prescribed values.

Lemma 2. Suppose two functions of s variables, $m(\lambda)$ and $M(\lambda)$, and a number h are given such that for any $\lambda^{(0)}$ there is a number $\lambda_{j+1}^{(0)}$ such that the cube with center at $(\lambda^0, \lambda_{j+1}^{(0)})$ and with side length h is entirely contained between $m(\lambda)$ and $M(\lambda)$

$$E(\lambda : |\lambda_j - \lambda_j^{(0)}| \leqq h/2) \subset E(\lambda : m(\lambda) < \lambda_{j+1} < M(\lambda)).$$

Then there exists a trigonometric polynomial $P(\lambda)$ such that

$$m(\lambda) < P(\lambda) < M(\lambda).$$

Lemma 3. Suppose a function $A(\lambda) \in \mathfrak{p}$ of s variables has a skeleton, which is positive in the main, except possibly for an s-dimensional element, and satisfies the condition

$$D^\gamma A^{(\rho)}(0,\mu) < D^\gamma X^{(\rho)}(0,\mu), \qquad X(\lambda) > 0 \quad \text{for} \quad \lambda_1, \lambda_2, \dots, \lambda_n \neq 0 \tag{17}$$

for all γ such that

$$\mathfrak{A}^{(\rho)}(\gamma) \subset Sc\{A(\lambda)\}. \tag{18}$$

The function $X(\lambda)$ may be called a skeleton majorant for $A(\lambda)$.

Then for any $\gamma^* \in Mc\{A\}$ there is a function $\omega(\lambda)$ such that

$$A_1(\lambda) = A(\lambda) + \sin^{\gamma^*} \frac{\lambda}{2}\, \omega(\lambda),$$

will have a skeleton positive in the main, and inequality (17) will remain valid.

Theorem 1 is proved by constructing $A_1(\lambda)$ step by step, beginning with zero-dimensional traces and ending with n-dimensional ones.

Theorem 2. If the function $K(q)$, periodic with period matrix H, satisfies all the remaining conditions of the theorem in the fundamental parallelepiped, then it can be represented as a sum of squares of functions with periods twice as large.

The proof is based on the change of variables $q = H^{-1}\lambda + Q(\lambda)$, where $Q(\lambda)$ is a periodic vector function with period matrix H, with the first terms of the expansion in powers of λ chosen in the required manner.

Corollary. If the function $K(p)$ is analytic, real, and periodic with period matrix H, is positive everywhere except at the origin, and the leading terms of its expansion are positive and have even order in all variables, then this function can be represented as a sum of squares.

From this follows the possibility of representations in the form (1) of a difference operator inverse to the discrete polyharmonic potential considered by us in notes [1,2], and consequently another method of expanding the corresponding difference operator, and another proof of the theorem on the asymptotic optimality of cubature formulas (see [3]).

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[3] S. L. Sobolev, *Lectures on the Theory of Cubature Formulas*, Novosibirsk, 1965.

Note: Figure translations are in progress. See original paper for figures.