MATHEMATICS

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1965

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Abstract

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MATHEMATICS

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ON A QUESTION IN THE THEORY OF WATSON TRANSFORMATIONS

(Presented by Academician I. N. Vekua on 16 III 1965)

For Watson transformations the following fundamental theorem is known (see ([1]), pp. 294–295).

Let the functions $K(s)$ and $H(s)$ satisfy the conditions:

$$\sup_{\operatorname{Re} s=1/2} \{|K(s)|,\ |H(s)|\} < \infty, \tag{1}$$

$$K(s)H(1-s) = 1, \qquad \text{when } \operatorname{Re} s = 1/2, \tag{2}$$

and let $x^{-1}k(s)$ and $x^{-1}h(x)$ be, respectively, the originals of the functions

$$\frac{K(s)}{1-s}, \quad \frac{H(s)}{1-s} \in L_2(1/2 - i\infty,\ 1/2 + i\infty) \tag{3}$$

under the Mellin transform (the originals of the Mellin transform will be called simply originals) (see, for example, ([1]), pp. 126–127).

Then:

1. For every function $f(x) \in L_2(0,\infty)$, the transformations

$$g^{(h)}(x) = \frac{d}{dx}\int_0^\infty \frac{h(xu)}{u} f(u)\,du, \qquad g^{(k)}(x) = \frac{d}{dx}\int_0^\infty \frac{k(xu)}{u} f(u)\,du \tag{4}$$

define functions from $L_2(0,\infty)$, by which $f(x)$ is determined almost everywhere on $(0,\infty)$ by the equalities

$$f(x) = \frac{d}{dx}\int_0^\infty \frac{h(xu)}{u} g^{(k)}(u)\,du, \qquad f(x) = \frac{d}{dx}\int_0^\infty \frac{k(xu)}{k} g^{(h)}(u)\,du. \tag{5}$$

2. Parseval' s equality holds:

$$\int_0^\infty g^{(h)}(x)g^{(k)}(x)\,dx = \int_0^\infty |f(x)|^2\,dx. \tag{6}$$

In the present note the problem is posed of clarifying to what extent condition (2) is essential for each of the two assertions of the theorem, taken separately.

Let the kernels of the Watson transformations have the representations

$$H(s) = H_1(s) + H_2(s), \qquad K(s) = K_1(s) + K_2(s), \tag{7}$$

where

$$\sup_{\operatorname{Re} s = 1/2} \{|H_{1,2}(s)|,\ |K_{1,2}(s)|\} < \infty; \tag{8}$$

$x^{-1}h_{1,2}(x)$, $x^{-1}k_{1,2}(x)$ are, respectively, the originals of the functions $(1-s)^{-1}H_{1,2}(s)$, $(1-s)^{-1}K_{1,2}(s)$.

Theorem. *For every function $f(x) \in L_2(0,\infty)$ and functions $h_{1,2}(x)$, $k_{1,2}(x)$ satisfying the above conditions, the transformations*

$$g_{1,2}^{(h)}(f,x) = \frac{d}{dx}\int_0^\infty \frac{h_{1,2}(xu)}{u} f(u)\,du, \qquad g_{1,2}^{(k)}(f,x) = \frac{d}{dx}\int_0^\infty \frac{k_{1,2}(xu)}{u} f(u)\,du \tag{9}$$

define functions of the class $L_2(0,\infty)$; moreover:

1) if

$$K_1(s)H_1(1-s) + K_2(s)H_2(1-s) = 1, \quad \text{when } \operatorname{Re} s = \tfrac{1}{2}, \tag{10}$$

then almost everywhere on $(0,\infty)$ the equality

$$f(x) = \frac{d}{dx}\int_0^\infty \frac{k_1(xu)}{u}\, g_1^{(h)}(u)\,du + \frac{d}{dx}\int_0^\infty \frac{k_2(xu)}{u}\, g_2^{(h)}(u)\,du; \tag{11}$$

holds;

2) if not only condition (10) is satisfied, but also the condition

$$K_1(s)H_2(1-s) + K_2(s)H_1(1-s) = A(s), \tag{12}$$

where

$$\inf_{\operatorname{Re} s=1/2} |1 + A(s)| > 0, \tag{13}$$

then almost everywhere on $(0, \infty)$ the equality

$$\int_0^\infty g^{(k)}(\tilde{f}, x)\, g^{(h)}(f, x)\, dx = \int_0^\infty |f(x)|^2\, dx, \tag{14}$$

holds, where $k(x) = k_1(x) + k_2(x)$ $(h(x) = h_1(x) + h_2(x))$; $\tilde{f}(x)$ is the original of the function

$$\widetilde{F}(s) = [1 + A(1-s)]^{-1} F(s). \tag{15}$$

Proof. The validity of assertion (11) is established by a simple repetition of the corresponding steps in the proof of Watson' s theorem as applied to the equality

$$F(s) = K_1(s) G_1^{(h)}(1-s) + K_2(s) G_2^{(h)}(1-s), \tag{10'}$$

$$\operatorname{Re} s = \tfrac{1}{2}, \qquad G_{1,2}^{(h)}(s) = H_{1,2}(s) F(1-s).$$

To prove the second assertion of the theorem, denote

$$\widetilde{K}_{1,2}(s) = [1 + A(s)] K_{1,2}(s), \qquad \widetilde{K}(s) = \widetilde{K}_1(s) + \widetilde{K}_2(s), \tag{16}$$

and the originals of the functions $(1-s)^{-1}\widetilde{K}(s)$, $(1-s)^{-1}\widetilde{K}_{1,2}(s)$ respectively by $x^{-1}\tilde{k}(x)$, $x^{-1}\tilde{k}_{1,2}(x)$. We now note that

$$\widetilde{K}(s) H(1-s) = 1, \qquad \operatorname{Re} s = \tfrac{1}{2}; \tag{17}$$

therefore it follows from Watson' s theorem that

$$\int_0^x f(x)\, dx = \int_0^\infty \frac{\tilde{k}(xu)}{u} [g_1^{(h)}(f, u) + g_2^{(h)}(f, u)]\, du =$$

$$= \int_0^\infty \frac{\tilde{k}(xu)}{u} g^{(h)}(f, u)\, du, \qquad x > 0; \tag{18}$$

$$\int_0^\infty g^{(\tilde{k})}(f, x)\, g^{(h)}(f, x)\, dx = \int_0^\infty |f(u)|^2\, du, \tag{19}$$

where

$$g^{(\tilde{k})}(f,x) = g_1^{(\tilde{k})}(f,x) + g_2^{(\tilde{k})}(f,x), \tag{20}$$

and $g_{1,2}^{(\tilde{k})}(f,x)$ is defined as in (9), but for the functions $\tilde{k}_{1,2}(x)$. Knowing that, according to (13) and (15), $F(s) \in L_2(1/2-i\infty,\ 1/2+i\infty)$, we form the function

$$I(x) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{K(s)x^{1-s}}{1-s}\,\frac{F(1-s)}{1+A(s)}\,ds, \qquad x > 0. \tag{21}$$

Using the convolution theorem for the Mellin transform, from (20) we obtain

$$I(x) = \int_0^\infty \frac{k(xu)}{u}\,\tilde{f}(u)\,du \tag{22}$$

or

$$I(x) = \int_0^x g^{(k)}(\tilde{f},x)\,dx. \tag{22'}$$

On the other hand, from (21), in accordance with (16), we have

$$I(x) = \int_0^\infty \frac{\tilde{k}(xu)}{u}\,f(u)\,du = \int_0^x g^{(\tilde{k})}(f,x)\,dx. \tag{22''}$$

From (22′) and (22″) it follows that almost everywhere on $(0,\infty)$

$$g^{(\tilde{k})}(f,x) = g^{(k)}(\tilde{f},x). \tag{23}$$

The theorem is proved.

Remark 1. In the theorem the functions $k_{1,2}(x)$, $h_{1,2}(x)$ ($k(x), h(x)$) have equal rights; therefore its assertions remain valid also when k and h are interchanged in them.

Remark 2. The theorem can also be extended to the case when the functions $H(s)$ and $K(s)$ are represented as sums of more than two terms, with, of course, the imposition of the corresponding restrictions similar to (10), (12), (13).

Corollary 1. In the case when, for $\operatorname{Re} s = 1/2$,

$$A(s) = 0, \qquad \tilde{k}(x) = k(x),$$

the theorem reduces to Watson’ s theorem.

Corollary 2. In the case when, for $\operatorname{Re} s = 1/2$,

$$A(s) \neq 0, \qquad \inf |1 + A(s)| > 0,$$

the role of the Parseval equality for the transform is played by equality (14), which is in general different from (6).

Remark. From the theorem given in the article, not only do the transforms studied earlier in the works ([2,3]) follow as special cases, but an answer is also given to the question left open in them concerning the Parseval equality (in the sense of Watson transforms).

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Received
9 III 1965

CITED LITERATURE

1. E. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, M.–L., 1948.
2. M. M. Dzhrbashyan, *Izv. AN SSSR, Ser. Mat.*, **19**, 133 (1955).
3. M. M. Dzhrbashyan, *Izv. AN SSSR, Ser. Mat.*, **24**, No. 3, 386 (1960).

Note: Figure translations are in progress. See original paper for figures.