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Abstract

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ON THE CONVERGENCE OF FOURIER SERIES OF ALMOST PERIODIC FUNCTIONS

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1. This note sets forth a method that provides a number of new criteria for the uniform and absolute convergence of Fourier series of almost periodic (a.p.) functions. It is based on representing the given function as the sum of a finite number or a countable set of a.p. functions with bounded spectra, and is applicable mainly to the class Q of uniform a.p. functions whose Fourier exponents have a finite number of limit points on each segment of the real axis.

Let the Fourier series of the uniform a.p. function $f(x)$ be written in symmetric form:

$$f(x) \sim \sum_{k=-\infty}^{\infty} A_{\lambda_k} e^{i\lambda_k x} \qquad (\lambda_0 = 0, \quad \lambda_k = -\lambda_k; \quad \lambda_k > 0 \text{ for } k > 0). \quad (1)$$

Put $\|f\| = \mathrm{Sup}_x |f(x)|$; by $L(f)$ denote the set of absolute values of the Fourier exponents of the function $f(x)$. We shall call any $\sigma > 0$ an isolated point from the right with respect to the set $L(f)$, if there exists $\varepsilon > 0$ such that the interval $(\sigma, \sigma + \varepsilon)$ contains no points of the set $L(f)$. For $f(x) \in Q$ introduce into consideration the sequence $\bar{L} = \{\Lambda_j\}$ $(j = 1, 2, \dots;\ 0 < \Lambda_j < \Lambda_{j+1})$ of all finite nonzero limit points of the set $L(f)$. Let the numbers ε_j $(j = 0, 1, \dots)$ be chosen so that the intervals $I_j = (\Lambda_j - \varepsilon_j, \Lambda_j + \varepsilon_j)$ $(j = 0, \pm 1, \pm 2, \dots,\ \Lambda_{-j} = -\Lambda_j,\ \Lambda_0 = 0,\ \varepsilon_{-j} = \varepsilon_j)$ have no common points. By η_j denote the distance between the intervals I_j and I_{j+1} $(j \geq 0)$ (if $\Lambda_0 = 0$ is not a limit point of the set $L(f)$, then in defining η_0 we take $\varepsilon_0 = 0$, and subsequently we shall put $j \neq 0$).

Put $I = \bigcup_j I_j$ and let $M = L(f) \cap CI$, where CI is the complement of the set I.

The set $L_j(f) = L(f) \cap I_j$ may, without loss of generality, be regarded as symmetric with respect to Λ_j. The points λ of the set $L_j(f)$ satisfying the condition $\lambda - \Lambda_j > 0$ are renumbered in decreasing order; as a result we obtain the sequence

$$l^{(j)} = \{\lambda_k^{(j)} - \Lambda_j\} \quad (k = 1, 2, \dots;\ \Lambda_j < \lambda_{k+1}^{(j)},\ \lim_{k\to\infty} \lambda_k^{(j)} = \Lambda_j).$$

We shall assign the function $f(x) \in Q$ to the class Q_0 if its spectrum is bounded; we shall assign the function $f(x) \in Q$ to the class Q_1 if the Fourier exponents of the function $f(x)$ have a finite number of finite limit points and a limit point at infinity (in both cases the sequence $\bar{L}$ is finite). For $f(x) \in Q_1$, for sufficiently large $\sigma = \sigma(f)$, the set $L_\sigma(f) = L(f) \cap [\sigma, \infty)$ has a unique limit point at infinity; renumbering its terms in increasing order, we obtain the sequence

$$l^{(\sigma)} = \{\lambda_k^{(\sigma)}\} \quad (k = 1, 2, \dots;\ \sigma \le \lambda_k^{(\sigma)} < \lambda_{k+1}^{(\sigma)};\ \lim_{k\to\infty} \lambda_k^{(\sigma)} = \infty).$$

Let $l = \{a_k\}$ $(k = 1, 2, \dots)$ be a decreasing (increasing) sequence of positive numbers. We shall call this sequence θ-lacunary if there exists $\theta > 1$ such that $a_k/a_{k+1} \ge \theta$ $(a_{k+1}/a_k \ge \theta)$; we shall call it θ, r-lacunary if it admits a representation

$$l = \bigcup_{i=1}^{s} l_i, \quad \text{where } s \le r,\ l_i \text{ is a } \theta\text{-lacunary sequence}$$

(see ([3],[5],[6])).

Let $0 < \Lambda_1 < \Lambda_2 < \dots < \Lambda_m$ be the nonzero (finite) limit points of the set $L(f)$ of the function $f(x) \in Q_0$ $(f(x) \in Q_1)$. We assign $f(x) \in Q_0$ to the class $\mathcal{L}_0(\theta, r)$ if the sequences $l^{(j)}$ $(j = 0, \pm1, \pm2, \dots, \pm m)$ are θ, r-lacunary; we assign $f(x) \in Q_1$ to the class $\mathcal{L}_1(\theta, r)$ if the sequences $l^{(\sigma)} l^{(j)}$ $(j = 0, \pm1, \pm2, \dots, \pm m)$ are θ, r-lacunary.

Let $f(x) \in Q$ and let the sequence $\overline{L}$ be infinite; if the set M is infinite, renumber its elements in increasing order; as a result we obtain the sequence $l^{(M)} = \{\lambda_k(M)\}$ $(k = 1, 2, \dots)$. We assign $f(x)$ to the class $\mathcal{L}(\theta, r)$ if: 1) the sequences $\overline{L}$, $l^{(j)}$ $(j = 0, \pm1, \pm2, \dots)$ are θ, r-lacunary and, in the case when the set M is infinite, the sequence $l^{(M)}$ is also θ, r-lacunary; 2) there exist $\varepsilon > 0$ and $\eta < 0$ such that $\varepsilon_j \ge \varepsilon$, $\eta_j \ge \eta$ $(j = 0, 1, \dots)$.

2. Lemma 1. If $f(x) \in Q_0$, then

$$f(x) = P(x) + \sum_{j=-m}^{m} f_j(x), \tag{2}$$

where

$$P(x) = \sum_{\lambda_k \in I} A_{\lambda_k} e^{i\lambda_k x}, \qquad f_j(x) \in Q_0, \qquad f_j(x) \sim \sum_{\lambda_k \in I_j} A_{\lambda_k} e^{i\lambda_k x};$$

moreover, the estimate

$$\|f_j\| \le \left[\frac{4}{\pi} + \frac{2}{\pi} \ln\left(1 + \frac{2\varepsilon}{\eta}\right) + 2N\right] \|f\|, \tag{3}$$

holds, where $\varepsilon = \max_j\{\varepsilon_j\}$, $\eta = \min_j\{\eta_j\}$, and N is the number of points belonging to the set M.

We shall write the Fourier series of the functions $f_j(x)$ $(j = 0, \pm1, \pm2, \dots, \pm m)$ and $f(x) \in Q_0$ in the following form:

$$f_j(x) \sim \sum_{k=-\infty}^{\infty} A_{\lambda_k^{(j)}} e^{i\lambda_k^{(j)} x}$$

$$\Big(\lambda_j < \lambda_{k+1}^{(j)} < \lambda_k^{(j)} \text{ for } k \geq 0; \quad \lim_{k\to\infty} \lambda_k^{(j)} = \Lambda_j; \quad \lambda_{-k}^{(j)} = 2\Lambda_j - \Lambda_k^{(j)};$$

$$|A_{\lambda_k^{(j)}}| + |A_{\lambda_{-k}^{(j)}}| > 0 \text{ for } k > 0\Big), \tag{4}$$

$$f(x) \sim P(x) + \sum_{k=-\infty}^{\infty} \sum_{j=-m}^{m} A_{\lambda_k^{(j)}} e^{i\lambda_k^{(j)} x}. \tag{1'}$$

Let σ be a point isolated on the right with respect to the set $L(f)$, $\mu > \sigma$; enclose all points of the set $L_{\sigma,\mu} = L(f) \cap (\sigma, \mu]$ in disjoint intervals $\delta_i = [\alpha_i, \beta_i]$ $(\sigma < \alpha_i \leq \beta_i \leq \mu,\ i = 1, 2, \dots, n;\ n \geq 1)$, renumbered in increasing order of their left endpoints. Choose numbers $\nu_i > 0$ so that the intervals $\Delta_i = (\alpha_i - \nu_i, \beta_i + \nu_i]$ $(i = 1, 2, \dots, n)$ do not intersect and belong to the semi-interval $(\sigma, \mu]$ (when $\beta_n = \mu$, the last of the intervals is replaced by the semi-interval $(\alpha_n - \nu_n, \beta_n])$. By $\Delta_f = \Delta_f(\sigma, \mu)$ denote any system of intervals $\Delta_i \supset \delta_i$ $(i = 1, 2, \dots, n)$ covering the set $L_{\sigma,\mu}$.

Lemma 2. Let $f(x)$ be a uniformly a.p. function, and let σ be a point isolated on the right with respect to the set $L(f)$; then there exists a uniformly a.p. function

$$f_\sigma(x) \sim \sum_{|\lambda_k| \leq \sigma} A_{\lambda_k} e^{i\lambda_k x}$$

and the estimate

$$\|f - f_\sigma\| \leq \Phi(\sigma, \mu, \Delta_f) A_\sigma(f), \tag{5}$$

holds.

where $A_\sigma(f)$ is the best approximation of the function $f(x)$ by entire functions of degree $\leqslant \sigma$,

$$\Phi(\sigma, \mu, \Delta_f) = 1 + \frac{4}{\pi} + \frac{2}{\pi} \ln \frac{\mu + \sigma}{|\mu - \sigma|} + \frac{1}{\pi(\mu - \sigma)}$$

$$\cdot \int_{-\infty}^{\infty} \left| \sum_{\alpha_i \neq \beta_i} \left[\frac{\beta_i - \alpha_i}{\nu_i} \frac{\cos(\alpha_i - \nu_i)u - \cos\alpha_i u}{u^2} - \frac{\cos\alpha_i u - \cos\beta_i u}{u^2} \right] \right.$$

$$\left. + \sum_{\beta_i \leqslant \mu} \frac{\mu - \beta_i}{\nu_i} \frac{\cos(\alpha_i - \nu_i)u - \cos\alpha_i u - \cos\beta_i u + \cos(\beta_i + \nu_i)u}{u^2} \right| du.$$

The proof of Lemmas 1 and 2 is based on the integral representation (1) of a.p. functions $f_j(x)$ and $f_\sigma(x)$ with kernels that are linear combinations of the kernels of N. I. Akhiezer–B. M. Levitan ([2,7]).

Lemma 3. If for a uniformly a.p. function $f(x)$ there exist an increasing sequence $\{\sigma_k\}$ ($k = 1, 2, \ldots;\ \lim_k \sigma_k = \infty$) of points isolated on the right with respect to the set $L(f)$, a sequence $\{\mu_k\}$ ($k = 1, 2, \ldots;\ \mu_k > \sigma_k$), and a sequence of systems of intervals

$$\Delta_f^{(k)} \quad (k = 1, 2, \ldots;\ \Delta_f^{(k)} = \Delta_f(\sigma_k, \mu_k))$$

such that

$$\sum_{k=1}^{\infty} \Phi(\sigma_k, \mu_k, \Delta_f^{(k)}) A_{\sigma_k}(f) < \infty,$$

then, uniformly on the entire real axis,

$$f(x) = \sum_{k=1}^{\infty} f_k(x),$$

where

$$f_k(x) = f_{\sigma_k}(x) - f_{\sigma_{k-1}}(x) \quad (k = 2, 3, \ldots), \qquad f_1(x) = f_{\sigma_1}(x);$$

moreover,

$$\sum_{k=1}^{\infty} \|f_k\| < \infty.$$

3. In what follows it is assumed that the limit points of the set of Fourier exponents of the function $f(x)$ do not belong to the spectrum; however, this restriction can be removed without substantially complicating the proofs. Lemma 1 makes it possible to extend criteria for uniform and absolute convergence of Fourier series of a.p. functions whose Fourier exponents have a single limit point $\Lambda^* \neq \infty$ to functions of the class Q_0. We shall confine ourselves in this direction to the formulation of two theorems.

Theorem 1. If for $f(x) \in Q_0$ there exist constants $a > 0$ and $C > 0$ such that

$$\left|\frac{1}{u}\int_x^{x+u} f(t)e^{-i\lambda_j t}\,dt\right| < \frac{C}{|u^\alpha|} \qquad (0 < \alpha \leqslant 1;\ j = 0, \pm 1, \dots, \pm m), \tag{6}$$

$$N_j\left(\frac{\varepsilon}{1 + a\varepsilon}\right) - N_j(\varepsilon) = O(1) \qquad (j = 0, \pm 1, \dots, \pm m), \tag{7}$$

where

$$N_j(\varepsilon) = \sum_{\varepsilon \leqslant |\lambda_k - \Lambda_j| \leqslant \varepsilon_j} 1,$$

then the series $(1')$ converges uniformly.

Proof. Conditions (6) and (7) imply, by ((6), Theorem 8), the uniform convergence of the series (4). By virtue of (2), the series $(1')$ converges uniformly.

Theorem 2. If $f(x) \in \Pi_0(\theta, r)$, then

$$\sum_{k=-\infty}^{\infty} |A_{\lambda_k}| \leqslant \widetilde{C}\|f\|, \tag{8}$$

where

$$\widetilde{C} = (2m + 1)C(\theta, r)\left[\frac{4}{\pi} + \frac{2}{\pi}\ln\left(1 + \frac{2\varepsilon}{\eta}\right) + 2N\right] + N$$

and the constant $C(\theta, r)$ depends only on θ and r.

Proof. By virtue of Theorem 7 of paper [6],

$$\sum_{k=-\infty}^{\infty} \left|A_{\lambda_k^{(j)}}(i)\right| \leq C(\theta, r)\|f_j\|;$$

applying (2) and (3), we obtain (8). Let Q_∞ be the class of uniformly a.p. functions whose Fourier exponents have a unique limit point at infinity. For $f(x) \in Q_1$, Lemma 2 implies the equality $f(x) = f_\sigma(x) + \tilde{f}_\sigma(x)$, where $f_\sigma(x) \in Q_0$, $\tilde{f}_\sigma(x) \in Q_\infty$; it allows one to extend to the class Q_1 the criteria for convergence of Fourier series known for the classes Q_0 and Q_1 [1,4–6]. We give an example of such a generalization.

Theorem 3. *If $f(x) \in \mathcal{L}_1(\theta, r)$, then the series (1) converges absolutely.*

The proof follows from Theorem 2 of the present note and Theorem 9 of paper [5].

On the basis of Lemma 3 one can obtain a number of criteria for absolute convergence of Fourier series in the case when the set $L(f)$ is not everywhere dense. In particular, for $f(x) \in \mathcal{L}(\theta, r)$, in estimate (8), applied to the functions

$f_k(x) \in \mathcal{L}_0(\theta, r)$ $(k = 1, 2, ...)$, the constants $\tilde{C}$ are uniformly bounded, and it is possible to choose sequences $\{\sigma_k\}$, $\{\Delta_f^{(k)}\}$ $(k = 1, 2, \ldots;\ \sigma_k \geq 2^k)$ such that $\Phi(\sigma_k, 2\sigma_k, \Delta_f^{(k)}) = O(1)$. Therefore the following holds.

Theorem 4. *The Fourier series of a function $f(x) \in \mathcal{L}(\theta, r)$ converges absolutely if $f(x) \in \mathrm{Lip}\, a$ $(a > 0)$.*

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Note: Figure translations are in progress. See original paper for figures.