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Abstract

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ON SUBGROUPS OF THE MULTIPLICATIVE GROUP OF A DIVISION RING

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Let D be an arbitrary division ring. We shall denote the multiplicative group of this division ring by D^*; the center of the division ring D by Z; the multiplicative group of the field Z by Z^*; and the identity element of the group D^* by e.

Herstein ([5]) proved some theorems on finite subgroups of the group D^*. Amitsur ([4]) described all finite groups that can be embedded in the multiplicative group of some division ring. In particular, every finite p-subgroup of the group D^* is either a cyclic group or a group of generalized quaternions.

D. A. Suprunenko informed the author that for $p \neq 2$ the Sylow p-subgroups of the multiplicative group of a division algebra finite-dimensional over its center are conjugate, while Sylow 2-subgroups are conjugate in a number of special cases. In particular, in order to prove the conjugacy of Sylow 2-subgroups of the multiplicative group of a finite-dimensional division algebra it is necessary to show that, if one of the Sylow 2-subgroups is a group of generalized quaternions with defining relations $a^{2^n} = 1$, $b^2 = a^{2^{n-1}}$, $b^{-1}ab = a^{-1}$, then any other Sylow 2-subgroup can be neither a cyclic group of order greater than 2^n nor a group of type 2^∞.

We shall construct an example of a division ring D whose multiplicative group D^* contains, as Sylow 2-subgroups, a cyclic group F_1 of order 8 and a group F_2 of generalized quaternions. The division ring D will have dimension 16 over its center Z, which is the field of rational functions $R(t)$ in one variable over the field of rational numbers R.

Let $R(x)$ be the field of rational functions over R; let s be the automorphism of the field $R(x)$ generated by the mapping $x \to -x^{-1}$; let $A = R(x)[Y, s]$ be the set of noncommutative polynomials over $R(x)$ of the form $a_0 + a_1 Y + ... + a^m_{mY}$, where $a_i \in R(x)$, Y is an indeterminate, and multiplication and addition are defined in the natural way by means of the relation

$$(Y^k \alpha)(Y^l \beta) = Y^{k+l}(\alpha s^l)\beta.$$

It is known that A is an associative ring. It is clear that the element Y^2 lies in the center of A. It is not difficult also to see that the quotient ring Q of the ring A by the principal ideal $B = (Y^4 + 1)A$ contains a subfield, isomorphic to

$R(x)$, which we shall also denote by $R(x)$. We denote by a the residue class corresponding to the element Y under the natural homomorphism of A onto Q. Denote by R_1 the field $R(t)$, where $t = x^{-1}a^2 + xa^{-2}$. The element t is transcendental over R. It is not difficult to show that $(Q : R_1) = 8$.

Lemma 1. Let $t_1 = t - 2$, $t_2 = t + 2$. Denote by ρ_1, ρ_2 the p-adic norms of the field R_1 determined by the elements t_1, t_2, respectively. Let $\overline{R}_1$ be the completion of the field R_1 with respect to the norm ρ_1; let $\overline{\overline{R}}_1$ be the completion of the field R_1 with respect to the norm ρ_2, $\overline{Q} = Q_{R_1} \otimes \overline{R}_1$, $\overline{\overline{Q}} = Q_{R_1} \otimes \overline{\overline{R}}_1$. Then the algebras $\overline{Q}$ and $\overline{\overline{Q}}$ are division algebras of dimension 8 over the fields $\overline{R}_1$ and $\overline{\overline{R}}_1$, respectively.

Consider the subgroup G_1 of the group Q^* generated by the elements a and x. Each element g of the group G_1 is uniquely represented in the form $g = a^i x^j$, where $0 \leq i < 8$, and j is arbitrary. Let φ be a mapping of the group G_1 into itself,

constructed as follows: $\varphi(e) = e$, $\varphi(x) = x^{-1}$, $\varphi(a) = ax^{-1}$, $\varphi(a^i x^j) = \varphi(a)^i \varphi(x)^j$. It can be shown that φ is an automorphism of the group G_1. The automorphism φ extends to an automorphism of the algebra Q over R_1, and then to an automorphism of $\overline{Q}$ over $\overline{R}_1$, which we shall also denote by φ.

Consider the polynomial ring $\overline{Q}[Z, \varphi]$, where $\alpha Z = Z(\alpha\varphi)$ for $\alpha \in \overline{Q}$. Denote by Q_1 the quotient ring of the ring $\overline{Q}[Z, \varphi]$ by the principal ideal $(Z^2 + 1)$ (it is easy to see that the polynomial $Z^2 + 1$ lies in the center of $\overline{Q}[Z, \varphi]$).

Lemma 2. *The ring Q_1 is a division algebra of dimension 16 over the field $\overline{R}_1$.*

Denote by b the image of the element Z under the canonical homomorphism $\overline{Q}[Z, \varphi]$ onto Q_1. Let G be the group generated by G_1 and b; D the linear span of the group G over R. It follows from Lemma 2 that D is a division algebra over R_1 of dimension 16. It is not hard to show that the center of D coincides with the field R.

The group D^* contains a cyclic subgroup F_1 of order eight, generated by the element a, and a quaternion subgroup F_2, generated by the elements a^2 and b.

Lemma 3. *The subgroups F_1 and F_2 of the group D^* are Sylow 2-subgroups.*

In the proof of Lemma 3 one essentially uses the fact that the algebra $\overline{Q}$ is a division algebra over $\overline{R}_1$.

As has already been noted, in our example the center of the division ring D is the field of rational functions over R_1.

Theorem 1. *Let D be a finite-dimensional division algebra over a field T of algebraic numbers. Then the Sylow p-subgroups of the group D^*, for a given p, are conjugate.*

From the results of ([2]) it follows that all solvable normal divisors of the group D^* lie in the center. Herstein ([7]) showed that all locally nilpotent normal divisors of the group D^* lie in the center. In ([3]) it is proved that in the factor group D^*/Z^* there are no nontrivial locally finite normal divisors. At the same time, in ([1]) an example is given of a division ring whose multiplicative group contains an infinite descending chain of invariant subgroups**.

Proposition 1. *Let D be a division algebra of dimension q^m over its center T, which is algebraic over the field R of rational numbers. Then, if $q^m \neq 4$, the group D^* contains a noncentral normal divisor F for which the intersection of all terms of the lower central series is equal to the identity subgroup.*

Proposition 1 is used to prove Theorem 2, from which, in particular, there follows a series of known facts about normal divisors of the multiplicative group of a division ring and some new results.

Theorem 2. *Let D be an arbitrary division ring, and let G be a normal divisor of the group D^*. If for any elements x, y of G there exists a number n (depending on x, y) such that all right-normed n-fold commutators of the subgroup G_1 generated by x and y have finite orders, then $G \subset Z$.*

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* Finite dimensionality of T over the field of rational numbers is not assumed.

** *Note in proof.* In ([8]) it is shown that all locally solvable and all periodic normal divisors of the group D^* are contained in the center.

Note: Figure translations are in progress. See original paper for figures.