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Abstract

Full Text

MATHEMATICS

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ON INTEGRAL EQUATIONS WITH A KERNEL DEPENDING ON THE ABSOLUTE VALUE OF THE DIFFERENCE OF THE ARGUMENTS, AND WITH A FINITE INTERVAL OF VARIATION OF THE VARIABLES

1. In the paper (^{1}) with the same title we set forth certain considerations concerning the connection that exists between the solutions of equations of the form

$$\int_0^h \psi(\xi, h) K(|x-\xi|)\, d\xi = f(x), \tag{1}$$

where $h > 0$ is a finite constant, and $K(t)$ and $f(t)$ are prescribed functions, and the solutions of analogous equations with $h = \infty$ (a semi-infinite interval). The aim of the present paper is to reveal certain general properties of the solutions of equations of the form (1), especially of the equations called key equations in (^{1}).

2. We begin with the consideration of that special form of equations (1) for which $f(x) = -K(R + x)$, i.e.

$$\int_0^h \varphi(R, \xi, h) K(|x-\xi|)\, d\xi = -K(R + x), \qquad 0 < x < h, \tag{2}$$

where R and h are certain given constants.*

We shall study the solution $\varphi(R, \xi, h)$ of this equation as a function of all three variables R, ξ, h, assuming that R and h may vary within certain limits $0 < R \leq R_m,\ 0 < h \leq h_m$.

Replacing in (2) x by $x_1 = h - x$, R by η, and ξ by $\xi_1 = h - \xi$, we obtain the relation

$$\int_0^h \varphi(\eta, h-\xi, h) K(|x-\xi|)\, d\xi = -K(\eta + x_1), \qquad 0 < x < h, \tag{3}$$

and, replacing there h by $h_1 = h + \delta \le h_m,\ \delta > 0$,

$$\int_0^{h_1} \varphi(R, \xi, h_1) K(|x - \xi|)\, d\xi = -K(R + x), \qquad 0 < x < h + \delta. \tag{4}$$

For $0 < x < h$, (4) can be rewritten as

$$\int_0^{h} \varphi(R, \xi, h_1) K(|x - \xi|)\, d\xi =$$

$$= -K(R + x) - \int_0^{\delta} \varphi(R, h + \eta, h_1) K(\eta + x_1)\, d\eta. \tag{5}$$

Hence, taking (2) and (3) and the remark to formula (2) into account, we obtain

$$\varphi(R, \xi, h_1) = \varphi(R, \xi, h) + \int_0^{\delta} \varphi(R, h + \eta, h_1) \varphi(\eta, \xi_1, h)\, d\eta. \tag{6}$$

On the other hand, for $\delta < x < h_1$ one can rewrite (4) also as

$$\int_\delta^{h+\delta} \varphi(R, \xi, h_1) K(|x - \xi|)\, d\xi = -K(R + x) - \int_0^{\delta} \varphi(R, \xi, h_1) K(x - \xi)\, d\xi,$$

whence it is not difficult to obtain, similarly to the case of equations (5), (6), that for $\xi > \delta$

$$\varphi(R, \xi, h + \delta) = \varphi(R + \delta, \xi - \delta, h) + \int_0^{\delta} \varphi(R, \eta, h_1) \varphi(\delta - \eta, \xi - \delta, h)\, d\eta. \tag{7}$$

Subtracting (6) from (7), putting $\eta = \delta\tau$, and dividing by δ, we arrive at the relation

$$\frac{\varphi(R + \delta, \xi - \delta, h) - \varphi(R, \xi, h)}{\delta} = \int_0^1 \{\varphi(R, h + \delta\tau, h + \delta) \varphi(\delta\tau, \xi_1, h) -$$

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* Here, as everywhere below, we assume that the solutions of all the equations under consideration exist, are unique, and possess the properties required for the validity of the arguments carried out. In particular, we shall assume that the homogeneous equation obtained from (1) for $f(x) \equiv 0$ has only the solution $\psi(\xi, h) = 0$.

$$-\varphi(R, \delta\tau, h+\delta)\varphi(\delta[1-\tau], \xi-\delta, h)\}\, d\tau \equiv A, \tag{8}$$

whence, in particular, as $\delta \to 0$:

$$\partial\varphi(R, \xi, h)/\partial R - \partial\varphi(R, \xi, h)/\partial\xi = \lim_{\delta=0} A. \tag{9}$$

3. In order to proceed further, suppose that, from physical or other considerations, it is known that the function $\varphi(R, \xi, h)$ has the form

$$\varphi(R, \xi, h) = \frac{R^n}{[\xi(h-\xi)]^m} u(R, \xi, h), \tag{10}$$

where m and n are certain constants, of which m is known, satisfies the conditions $0 \le m < 1$, and is independent of h, while the function $u(R, \xi, h)$ does not tend to infinity unless R and ξ are simultaneously equal to zero.*

The expression for A now takes the form $A = R^n \delta^{n-m} B$, where B, as $\delta \to 0$, tends to the finite limit equal to

$$\lim_{\delta=0} B = \frac{1}{(h\xi\xi_1)^m}[u(R,0,h)u(0,\xi,h) - u(R,h,h)u(0,\xi_1,h)] \int_0^1 \frac{\tau^n\, d\tau}{(1-\tau)^m}. \tag{11}$$

Therefore the value of the quantity $\lim_{\delta=0} A$ depends first of all on the exponent $(n-m)$ of the power in which δ enters into A. But since the left-hand side of formula (9), on the one hand, is certainly finite for all $0 < \xi < h$ and $R \neq 0$, while, on the other hand, it cannot be identically equal to zero,** it must be that $n - m = 0$, and (9) assumes the form

$$\partial\varphi(R, \xi, h)/\partial R - \partial\varphi(R, \xi, h)/\partial\xi = F(R, \xi, h), \tag{12}$$

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* Thus, in electrostatic problems where $\varphi(R, \xi, h)$ corresponds to the density of surface charges induced by an external field on infinitely thin plates, $m = 1/2$; in diffraction problems, where $\varphi(R, \xi, h)$ is the density of induced currents, likewise $m = 1/2$, etc. (cf., for example, (2)). We also note that formula (10) could have been written without the factor R^n, but then, as the subsequent consideration carried out in the text shows, such a factor would emerge by itself from the function $u(R, \xi, h)$.

** This is evident at least from the fact that when $\lim_{\delta=0} A = 0$, (9) turns into the homogeneous equation $\partial\varphi/\partial R - \partial\varphi/\partial\xi = 0$, whose integral has the form $\varphi = \varphi(R, \xi, h) = f(R+\xi)$, where f is the symbol for some function. Since as $\xi \to 0$, according to (10), φ must tend to infinity independently of the value

$R > 0$ when $m > 0$, the function $f(R)$, and together with it $f(R+\xi) = \varphi(R, \xi, h)$, would have to tend to infinity for any value of its argument greater than zero, which is impossible.

where

$$F(R, \xi, h) = a_m (R/h\xi\xi_1)^m [u(R, h, h)u(0, \xi_1, h) - u(R, 0, h)u(0, \xi, h)],$$
$$0 < \xi < h, \qquad 0 < R < R_m, \qquad a_m = m\pi/\sin m\pi, \tag{13}$$

whereas (10) becomes

$$\varphi(R, 0, h) = (R/\xi\xi_1)^m u(R, \xi, h). \tag{14}$$

Thus the right-hand side of equation (12) contains three functions $u(0, \xi, h)$, $u(R, 0, h)$, and $u(R, h, h)$, each of which already depends on only one variable (not counting h, which we may now regard as a constant).

The general integral of equation (12) has the form

$$\varphi(R, \xi, h) = f(R+\xi) + \int_a^R F(\zeta, R+\xi-\zeta, h)\, d\zeta, \tag{15}$$

where f denotes an arbitrary function and a an arbitrary constant. Hence it is seen that $\varphi(R, \xi, h)$ is expressed in terms of four functions $u(0, \eta, h)$, $u(R, 0, h)$ and $u(R, h, h)$, and $f(R+\xi)$, each depending on only one variable (h, as already stated above, is here regarded as a constant). A particular form of the integral (15), satisfying the condition $\varphi(0, \xi, h) = 0$ for $0 < \xi < h$, which follows from (14) when $m > 0$, and suitable for $R + \xi < h$, may be written as

$$\varphi(R, \xi, h) = \int_0^R F(\zeta, R+\xi-\zeta, h)\, d\zeta, \tag{16}$$

i.e. in the present case $f \equiv 0$.

4. Without being able to dwell here on the question of the actual determination of the functions $u(0, \xi, h)$, $u(R, 0, h)$, $u(R, h, h)$, we restrict ourselves to the following remarks:

a) Suppose that, in some way (by numerical methods, etc.), solutions have been found of the equations*

$$\int_0^h \varphi(R_0, \xi, h) K(|x-\xi|)\, d\xi =$$

$$= R_0^m \int_0^h \frac{u(R_0,\xi,h)}{(\xi\xi_1)^m} K(|x-\xi|)\,d\xi = -K(R_0+x), \tag{17}$$

$$\int_0^h \frac{\partial\varphi(R_0,\xi,h)}{\partial R_0} K(|x-\xi|)\,d\xi = -K'(R_0+x), \tag{18}$$

where R_0 is any one of the possible values of R. Then, according to (12), the quantity

$$\chi(\xi) = \left(\frac{h\xi\xi_1}{R_0}\right)^m \frac{F(R_0,\xi,h)}{\alpha_m} = u(R_0,h,h)u(0,\xi_1,h) - \\ - u(R_0,0,h)u(0,\xi,h), \tag{19}$$

is known for all $0 < \xi < h$, where the constants $u(R_0,h,h)$ and $u(R_0,0,h)$ are known from the solution of equation (17). Since at the same time the value of the function is also known,

$$\chi(\xi_1) = u(R_0,h,h)u(0,\xi,h) - u(R_0,0,h)u(0,\xi_1,h),$$

we obtain

$$u(0,\xi,h) = \frac{\chi(\xi)u(R_0,0,h) + \chi(\xi_1)u(R_0,h,h)}{u^2(R_0,h,h) - u^2(R_0,0,h)}, \tag{20}$$

(18) is obtained from (2) by differentiating with respect to R and then substituting $R = R_0$.

i.e., $u(0,\xi,h)$ is completely determined. Moreover, it can be shown that $\lim \xi u(0,\xi,h) = -mh^m/\alpha_m$.

When the function $u(0,\xi,h)$ has been found, one can, using formula (16) and the known values $\varphi(R_0,\xi,h)$ and $\partial\varphi(R_0,\xi,h)/\partial R_0$, also find the functions $u(R,0,h)$ and $u(R,h,h)$ for $0 < R \leq R_0$, at least if $R_0 \leq h/2$. After this, the interval of values of R for which these functions are known can, if necessary, be extended.

b) Substituting (14) into (6), multiplying by ξ^m, and passing to the limit $\xi = 0$, we obtain

$$u(R,0,h) = \left(\frac{h}{h+\delta}\right)^m u(R,0,h+\delta) - \int_0^\delta \frac{\eta^m u(R,h+\eta,h+\delta)}{(h+\eta)^m(\delta-\eta)^m} u(\eta,h,h)\,d\eta, \tag{21}$$

and multiplying (14) by ξ_1^m and passing to the limit $\xi = h$:

$$u(R,h,h) = -\int_0^\delta \frac{\eta^m u(R,h+\eta,h+\delta)}{(h+\eta)^m(\delta-\eta)^m} u(\eta,0,h)\,d\eta. \tag{22}$$

Adding and subtracting (21) and (22) and setting $u(R,0,h) \pm u(R,h,h) = p_\pm(R,h)$, we arrive at the equations:

$$p_\pm(R,h) = \left(\frac{h}{h+\delta}\right)^m u(R,0,h+\delta) \mp \int_0^\delta \frac{\eta^m u(R,h+\eta,h+\delta)}{(h+\eta)^m(\delta-\eta)^m} p_\pm(\eta,h)\,d\eta. \tag{23}$$

If the solution of the key problem (4) is known for width $h_1 = h+\delta$, then the function $u(R,\xi,h_1)$ entering (23) is known, and the integral equations (23) can serve to find the functions $p_\pm(\eta,h)$ for $0<\eta<\delta$; after that, in the case $R>\delta$, their values for $R \geq \eta > \delta$ are obtained by substituting the already found values $p_\pm(\eta,h)$ for $0<\eta<\delta$ into the right-hand side of relations (23). We note in passing that $\lim Ru(R,0,h) = mh^m/\alpha_m$.

5. We now turn to the question of solving equation (1), and suppose that the general solution of the analogous equation for a larger interval $h_1 = h+\delta$, $\delta>0$, is known, i.e.,*

$$\int_0^{h_1} \psi(\xi,h_1)K(|x-\xi|)\,d\xi = f(x), \qquad 0<x<h_1. \tag{24}$$

Then, subtracting (1) from (24) and applying the same considerations as in Sec. 2 in the derivation of formula (6), we easily find that

$$\psi(\xi,h) = \psi(\xi,h_1) - \int_0^\delta \psi(\eta+h,h_1)\varphi(\eta,\xi_1,h)\,d\eta, \tag{25}$$

where $\varphi(R,\xi,h)$ is the solution of equation (2). Thus, the solution of equation (1) is obtained in quadratures through the solution of equation (24) and the solution of equation (2).

6. In this paper we have considered only equations of the first kind with kernel $K(|x-\xi|)$. For equations of the second kind, completely analogous results are obtained.

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CITED LITERATURE

[1] G. A. Grinberg, DAN, **128**, No. 3, 450 (1959).
[2] G. A. Grinberg, DAN, **129**, No. 2, 295 (1959).

* For example, for the semi-infinite case or others.

Note: Figure translations are in progress. See original paper for figures.

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