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Abstract

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MATHEMATICS

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ON ALMOST-PERIODIC SOLUTIONS OF NONLINEAR SYSTEMS

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In the present note sufficient conditions are indicated in order that the module of Fourier exponents of an almost-periodic solution of a nonlinear almost-periodic system be contained in the module of the right-hand side. In addition, theorems are formulated on the almost-periodicity of bounded solutions of almost-periodic "monotone" systems (see Definition 3).

1. Let an n-dimensional vector system be given

$$\dot{x} = F(x, t), \tag{1}$$

where $F(x, t)$ satisfies the local theorems of existence and uniqueness. Suppose that $F(x, t)$ is almost-periodic in t, uniformly in the cylinder $C_\Gamma\{x \in \Gamma,\ t \in I\}$, with module of Fourier exponents M_Γ (here Γ is some closed, bounded set in R^n, and $I = (-\infty, +\infty)$).

We shall say that $x(t)$ is contained in the cylinder C_Γ if $x(t) \in \Gamma$, $t \in I$. Let system (1) have an almost-periodic solution $x(t)$ contained in C_Γ. We give some conditions sufficient for the module of Fourier exponents M of this solution to satisfy the inclusion

$$M \subseteq M_\Gamma. \tag{2}$$

Denote by C the space of all continuous n-dimensional vector-functions bounded on I.

Definition 1. A set $K \subset C$ will be called an **invariant class with respect to translations** if, from the fact that $x(t) \in K$, $h \in I$, it follows that $x(t+h) \in K$, and from the fact that $x(t+h_n) \to y(t)$ (uniformly on I), it follows that $y(t) \in K$. Obviously, the intersection of two invariant classes is again an invariant class.

Theorem 1. Let $x(t)$ be some almost-periodic solution of system (1) in C_Γ. If to any other solution $y(t)$ of this system contained in C_Γ one can assign some invariant class K (depending, generally speaking, on $y(t)$) in such a way that $x(t) \in K$ and $y(t) \in K$, then (2) is valid for $x(t)$.

Hence, in particular, we obtain:

Corollary 1. If the almost-periodic solution $x(t)$ is the only representative in C_Γ of some invariant class K, then (2) is valid for $x(t)$.

If here, as K, one takes the set K^n of all almost-periodic n-vector-functions, then we obtain:

Corollary 2. If $x(t)$ is the unique almost-periodic solution of system (1) contained in C_Γ (other solutions extendable in both directions, but no longer almost-periodic, may be contained in C_Γ), then (2) holds for $x(t)$. In particular, if the almost-periodic solution $x(t)$ is in fact the unique solution wholly contained in C_Γ, then (2) holds for it.

Corollary 3. If (1) is a system with almost-periodic convergence ([2]), then the "limiting" solution of this system satisfies the inclusion (2).

Along with system (1), let us consider the entire collection of systems

$$\dot{x} = L(x, t), \tag{3}$$

where $L(x, t) \in H\{F(x, t)\}$ ([3]), i.e. $L(x, t)$ is obtained by a uniform limiting transition in C_Γ from some sequence of translates $\{F(x, t + h_n)\}$.

We now give some definitions and a theorem due to Amerio.

Definition 2. A solution $x^*(t)$ contained in C_Γ is called **separated in** C_Γ if for any other solution $y(t)$ contained in C_Γ, for all $t \in I$ we have

$$\|x^*(t) - y(t)\| \geq \rho^* > 0,$$

where ρ^* depends only on $x^*(t)$. If $x^*(t)$ is the unique solution contained in C_Γ, then we shall also call it separated in C_Γ.

If, for any system (3), all solutions contained in C_Γ are separated, then there can be only a finite number l of them, independent of $L(x, t)$ from the set $H\{F(x, t)\}$.

Amerio' s theorem ([1]). *If all solutions contained in C_Γ of any system from (3) are separated, then they are all almost periodic.*

In particular, if any system (3) admits only one solution contained in C_Γ, then it is almost periodic.

Under the conditions of this theorem, in the general case one can say nothing about the relation between the module M_Γ of the right-hand side of system (1) and the modules M of the separated solutions. With the aid of Corollary 2 and Theorem 1, where as invariant classes we take the sets

$$K_1 = K_{\text{ap}}^{(n)} \cap K_d^{(n)}; \qquad K_2 = K_{\text{ap}}^{(n)} \cap K_s^{(n)}; \qquad K_3 = K_{\text{ap}}^{(n)} \cap K_i^{(n)}, \tag{4}$$

we indicate a condition sufficient for all separated solutions in Amerio' s theorem to satisfy (2). Here $K_{\text{ap}}^{(n)}$ is the set of all almost-periodic n-vector functions; K_d^n is the set of all continuous, bounded n-vector functions with a given diameter d

$$d = \sup_{t_1, t_2 \in I} \|x(t_1) - x(t_2)\|; \tag{5}$$

K_s^n is the set with given s

$$s = \sup_{t \in I} \|x(t)\|; \tag{6}$$

$K_i^{(n)}$ is the set with given i

$$i = \inf_{t \in I} \|x(t)\|. \tag{7}$$

Corollary 4. *If in Amerio' s theorem the number of separated solutions is $l = 1$, then (2) is valid for the separated solution.*

In the general case, if for each pair chosen from among the $l > 1$ separated solutions there is at least one of the characteristics (5), (6), or (7) which takes different numerical values on the two solutions of the chosen pair, then all l separated solutions satisfy the inclusion (2).

If in Theorem 1 and in all the corollaries the function $F(x, t)$ is considered "purely" periodic with least period T, then in all the stated assertions the almost-periodic solution $x(t)$ will be periodic with least—

with the smaller period $\overline{T} = T/q$, where q is some natural number ($q \geqslant 1$), i.e. a Massera harmonic ([4]).

From Corollary 1 and the definition of the invariant classes $K_1^{(n)}$, $K_2^{(n)}$, and $K_3^{(n)}$ we obtain that if system (1) in C_Γ has some almost-periodic solution $x(t)$ with module M, not contained in M_Γ, then in C_Γ there must be at least one more almost-periodic solution $y(t)$ with the same module M such that any of the characteristics (5), (6), and (7) of this solution is equal to the corresponding characteristic of the solution $x(t)$, i.e. solutions not satisfying relation (2) cannot exist "in isolation."

In the general case of n-dimensional systems, in order for the inclusion (2) to hold we had to impose certain "uniqueness" conditions in some invariant class. It turns out that in the case $n = 1$ no additional conditions are required; namely, the following assertion holds.

Theorem 2. *If the first-order equation*

$$\dot{x} = f(x, t), \tag{8}$$

where $f(x,t)$ satisfies the local existence and uniqueness theorems and is an almost-periodic function uniformly in the strip $C_\Gamma\{a \leqslant x \leqslant b,\ t \in I\}$, has in this strip an almost-periodic solution $x(t)$, then the module of its Fourier exponents is necessarily contained in the module M_Γ of the function $f(x,t)$.

For $n > 1$ Theorem 2 is false, since it is known ([4]) that already for $n = 2$ a periodic system may have subharmonics.

Corollary 5. *If in equation (8) $f(x,t)$ is a periodic function with least period T, then every almost-periodic solution is a harmonic.*

This assertion, due to Massera, is a simple consequence of Theorem 2.

2. Let a Lyapunov function $V(x) \in C^{(1)}(R^{(n)})$ be given, positive definite and unboundedly increasing at infinity, i.e. for every $A > 0$ there exists $B > 0$ such that

$$V(x) > A, \quad \text{if } \|x\| > B.$$

Suppose, moreover, that we have a vector function $F(x,t)$, continuous in the aggregate of its variables.

Definition 3. We shall say that $F(x,t)$ is a V-monotonically decreasing function with respect to x in C_Γ if for every pair of vectors x and $y \in \Gamma$ $(x \neq y)$, for all $t \in I$,

$$W(x, y, t) \equiv (\operatorname{grad} V(x - y),\ F(x,t) - F(y,t)) < 0. \tag{9}$$

V-monotone increase is defined analogously, with the sign of the inequality changed to the opposite. In what follows we shall consider V-monotone decrease and shall simply say V-monotonicity. For monotone increase everything is rephrased in the obvious way.

Theorem 3. *If in system (1) the function $F(x,t)$ is V-monotone with respect to x in C_Γ and almost-periodic with respect to t uniformly on the set Γ, then in the cylinder C_Γ there can be no more than one solution defined on all of I (a limiting solution). This solution is an almost-periodic function with module of Fourier exponents M contained in the module M_Γ of the function $F(x,t)$. If some other solution remains in C_Γ for all $t \geqslant t_0$, then it asymptotically approaches the limiting solution as $t \to +\infty$.*

Theorem 4. *Suppose that in system (1) the function $F(x,t)$ is V-monotone in the whole space R^n and almost-periodic with respect to t uniformly in each set $D_R\{V(x) \leqslant R\}$, with a module of Fourier exponents uniform in D_R*

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M_R**.** If system (1) has, in some cylinder $C_R\{x \in D_R,\ t \in I\}$, a solution $x(t)$ bounded on the entire axis, then this solution is almost-periodic with the Fourier exponent module M, contained in M_R. All other solutions are unbounded to the left and tend asymptotically to $x(t)$ as $t \to +\infty$, i.e., almost-periodic convergence (2) takes place.

For $V(x) = \frac{1}{2}\|x\|^2$, this theorem generalizes Opial' s theorem (5) to the case of n dimensions.

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Note: Figure translations are in progress. See original paper for figures.