Mathematics

G. P. CHEREPANOV

1965

SovietRxiv

Abstract

Full Text

Mathematics

G. P. CHEREPANOV

BOUNDARY-VALUE PROBLEMS WITH ANALYTIC COEFFICIENTS

(Presented by Academician P. Ya. Kochina on 21 X 1964)

In the present paper a method is proposed for solving certain boundary-value problems in the theory of functions of a complex variable with analytic coefficients; the method consists in analytically continuing the original boundary condition into the complex plane and in directly solving the resulting functional equation. The simplest examples are considered.

§ 1. Method of functional equations. Let, on a closed analytic curve L, $x = x_0(s)$, $y = y_0(s)$ ($z = x_0(s) + iy_0(s) = z_0(s)$, $z_0'(s) \neq 0$ on L, s a real parameter), which divides the complex z-plane into two domains D^+ and D^-, a boundary condition of the form be given

$$F\{f(s)\overline{f(s)}, a_1(s), \dots, a_n(s)\} = 0. \tag{1,1}$$

Here $f(s)$ is the value on L of the sought function, analytic in the domain D^-, except, possibly, for a finite number of points at which it may have poles; $a_1(s), \dots, a_n(s)$ are certain functions (some or even all of the $a_i(s)$ may be unknown in advance); $F\{f, \bar{f}, a_1, \dots, a_n\}$ is a certain given function of its variables. Denote by $s(z)$ the function inverse to the function $z = z_0(s)$.

Theorem. *Suppose that: 1) $a_1(s), \dots, a_n(s)$, $s(z)$, $F\{f, \bar{f}, a_1, \dots, a_n\}$ are analytic functions of their variables, having only isolated singular points (in particular, branch points); 2) a solution of the boundary-value problem (1,1) exists.*

Then: 1) the functional equation

$$F\{f[s(z)],\ \overline{f[s(z)]},\ a_1[s(z)], \dots, a_n[s(z)]\} = 0 \tag{1,2}$$

holds in the whole z-plane; 2) every solution of the functional equation (1,2) satisfies the boundary condition (1,1), and conversely; 3) the solution of the original problem belongs to the class of analytic functions having only isolated singular points.

The theorem is proved by means of analytic continuation. It is not difficult to generalize it to the class of functions $a_1(s), \dots, a_n(s)$ and $s(z)$ whose natural boundary is different from punctured points and is a certain domain. An analogous theorem is valid for a system of boundary conditions and several sought functions.

To solve the functional equation (1,2), it is natural to expand all analytic functions in series in a neighborhood of some point and reduce the problem to the solution of the resulting infinite system of equations for the unknown coefficients. In some cases the infinite system degenerates into a finite one, and then the solution of the original problem can be obtained in closed form. In other cases, the form of the solution can be anticipated from the structure of the functional equation itself.

§ 2. Examples. Let us consider some very simple problems illustrating the application of the method of functional equations. A characteristic feature

of the method being applied is that it does not guarantee the uniqueness of the solutions obtained.

1°. Suppose it is required to find functions $\varphi(\zeta)$ and $\omega(\zeta)$, analytic outside the unit circle, from the boundary condition

$$a_1[\omega(\zeta)]^2 + a_2\omega(\zeta)\overline{\omega(\zeta)} + a_3[\overline{\omega(\zeta)}]^2 = a_4\varphi(\zeta) + a_5\overline{\varphi(\zeta)} + a_6 \quad \text{for } |\zeta| = 1, \tag{2,1}$$

where $a_1, \dots, a_6$ are complex constants.

As $\zeta \to \infty$, the functions $\varphi(\zeta)$ and $\omega(\zeta)$ behave as follows:

$$\varphi(\zeta) = b_1 + O(\zeta^{-1}), \qquad \omega(\zeta) = O(\zeta). \tag{2,2}$$

Consider the functional equation

$$a_1[\omega(\zeta)]^2 + a_2\omega(\zeta)\overline{\omega(1/\zeta)} + a_3[\overline{\omega(1/\zeta)}]^2 = a_4\varphi(\zeta) + a_5\overline{\varphi(1/\zeta)} + a_6. \tag{2,3}$$

Expand all functions in series in a neighborhood of the point at infinity and show that all conditions of the problem can be satisfied by setting

$$\omega(\zeta) = c_1\zeta + c_2 + c_3/\zeta, \qquad \varphi(\zeta) = b_1 + b_2/\zeta + b_3/\zeta^2. \tag{2,4}$$

Substituting the functions $\omega(\zeta)$ and $\varphi(\zeta)$ according to (2,4) into equation (2,3) and equating coefficients of equal powers of ζ, for the unknown coefficients c_i and b_i we obtain the system of algebraic equations

$$\begin{gathered}
a_1 c_1^2 + a_2 c_1 \overline{c_3} + a_3 \overline{c_3}^2 = a_5 \overline{b_3}, \\
2a_1 c_1 c_2 + a_2(c_1 \overline{c_2} + c_2 \overline{c_3}) + 2a_3 \overline{c_2 c_3} = a_5 \overline{b_2}, \\
a_1(c_2^2 + 2c_1 c_3) + a_2(c_1 \overline{c_1} + c_2 \overline{c_2} + c_3 \overline{c_3}) + a_3(\overline{c_2}^2 + \overline{c_1 c_3}) \\
= a_4 b_1 + a_5 \overline{b_1} + a_6, \\
2a_1 c_2 c_3 + a_2(c_2 \overline{c_1} + c_3 \overline{c_2}) + 2a_3 \overline{c_1 c_2} = a_4 b_2, \\
a_1 c_3^2 + a_2 c_3 \overline{c_1} + a_3 \overline{c_1}^2 = a_4 b_3.
\end{gathered} \tag{2,5}$$

The system of 5 algebraic equations (2,5) is, generally speaking, solvable with respect to the 5 unknowns c_1, c_2, c_3, b_2, b_3. Under the only assumption that the expansion at infinity of either of the functions $\omega(\zeta)$ or $\varphi(\zeta)$ terminates at some term, it follows, generally speaking, that the functions $\omega(\zeta)$ and $\varphi(\zeta)$ have the form (2,4), since if higher powers of ζ are admitted in the expansions, then, in order to determine the unknown coefficients, one obtains (generally speaking, unsolvable) overdetermined systems of algebraic equations.

Thus the solution of the original problem (2,1), (2,2) has been reduced to the solution of the algebraic system (2,5).

2°. Suppose it is required to find a function $\omega(\zeta) = u + iv$, analytic outside the unit circle, from the boundary condition

$$a_1(u^2 + v^2) = a_2 uv + a_3 \quad \text{for } |\zeta| = 1, \tag{2,6}$$

where a_1, a_2, a_3 are real constants. As $\zeta \to \infty$, the condition $\omega(\zeta) = O(\zeta)$ holds.

Consider the functional equation equivalent to the boundary condition (2,6):

$$a_1 \omega(\zeta)\, \overline{\omega}\left(\frac{1}{\zeta}\right) = \frac{a_2}{4i} \left\{ [\omega(\zeta)]^2 - \left[\overline{\omega}\left(\frac{1}{\zeta}\right)\right]^2 \right\} + a_3. \tag{2,7}$$

Expanding $\omega(\zeta)$ in a series in a neighborhood of the point at infinity, it is not difficult to see that, if this expansion terminates at some term,

then the function $\omega(\zeta)$ necessarily has the form

$$\omega(\zeta) = c_0 \zeta + c_1/\zeta + c_2. \tag{2,8}$$

Substituting the function $\omega(\zeta)$ according to (2,8) into equation (2,7), for the unknown coefficients c_0, c_1, and c_2 we obtain the following (generally speaking, solvable) system of three algebraic equations

$$4i a_1 c_0 \bar{c}_1 = a_2(c_0^2 - \bar{c}_1^2),$$

$$2ia_1(c_0\bar{c}_2 + c_2\bar{c}_1) = a_2(c_0c_2 - \bar{c}_2c_1), \tag{2,9}$$

$$4ia_1(c_2\bar{c}_2 + c_0\bar{c}_0 + c_1\bar{c}_1) = a_2(c_2^2 + 2c_0c_1 - \bar{c}_2^2 - 2\bar{c}_0\bar{c}_1) + 4ia_3.$$

3°. Suppose it is required to find a function $\omega(\zeta)$, analytic in the exterior of the unit disk, from the condition

$$a_1(\zeta)\,\overline{\omega(\zeta)} + a_2(\zeta) = \frac{\overline{\omega'(\zeta)}}{\omega(\zeta)|\omega(\zeta)|} \quad \text{for } |\zeta| = 1, \tag{2,10}$$

$$\omega(\zeta) = -\omega(-\zeta), \qquad \omega(\zeta) = O(\zeta) \quad \text{as } \zeta \to \infty.$$

Here $a_1(\zeta)$ and $a_2(\zeta)$ are unknown functions, analytic outside the unit disk and satisfying condition 1) of the theorem.

As $\zeta \to \infty$,

$$a_1(\zeta) = O(\zeta^{-3}), \qquad a_2(\zeta) = O(1). \tag{2,11}$$

Problem (2,10), (2,11) was encountered in the solution of one concrete elasto-plastic problem (^{1}).

Consider the functional equation

$$a_1'(\zeta)\,\overline{\omega(1/\zeta)} + a_2(\zeta) = \frac{\overline{\omega(\zeta^{-1})}}{\omega(\zeta)\sqrt{\omega(\zeta)\,\overline{\omega(\zeta^{-1})}}}. \tag{2,12}$$

We seek the function $\omega(\zeta)$ in the form

$$\omega(\zeta) = c_0\zeta + \bar{P}_\nu(\zeta^{-1}), \tag{2,13}$$

where $P_\nu(x)$ is a polynomial of degree ν.

Substituting (2,13) into (2,12) and expanding all functions in a series in a neighborhood of the infinitely distant point, it is not difficult to see that $\nu = 3$, so that, if the function $\omega(\zeta)$ has the form (2,13) and satisfies the condition $\omega(\zeta) = -\omega(-\zeta)$, then it is necessarily equal to

$$\omega(\zeta) = c_0\zeta + \frac{c_1}{\zeta} + \frac{c_3}{\zeta^3}. \tag{2,14}$$

From the functional equation (2,12) it then follows that the function $\omega(\zeta)\overline{\omega(\zeta^{-1})}$ must have pairwise coincident zeros; hence from (2,14) the equality follows

$$c_1^2 = 4c_0c_3. \tag{2,15}$$

Finally, the solution of the functional equation (2,12) is written in the form

$$\omega(\zeta) = \frac{c_0}{\zeta^3}\left(\zeta^2 + \frac{c_1}{2c_0}\right)^2, \tag{2,16}$$

whereby between the functions $a_1(\zeta)$ and $a_2(\zeta)$ there must exist the relation

$$\frac{c_3}{\zeta}\left(\zeta^2 + \frac{c_1}{2c_3}\right)^2 a_1(\zeta) + a_2(\zeta) = \frac{c_3^{1/2}\zeta^4(\zeta^2 + c_1/2c_3)}{c_0^{3/2}(\zeta^2 + c_1/2c_0)^3}. \tag{2,17}$$

Institute of Mechanics
Academy of Sciences of the USSR

Received
17 X 1964

CITED LITERATURE

1. G. P. Cherepanov, *Prikl. matem. i mekh.*, **27**, no. 3, 428 (1963).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.