ENGEL ELEMENTS AND THE RADICAL IN PI-ALGEBRAS AND TOPOLOGICAL GROUPS

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Abstract

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ENGEL ELEMENTS AND THE RADICAL IN PI-ALGEBRAS AND TOPOLOGICAL GROUPS

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Following R. Baer and B. Plotkin, an element g of a group G will be called a **left Engel element** or **nil-element** if for every $x \in G$ there exists an integer $n = n(x, g)$ such that the iterated commutator $[x, g(n)] = e$, where e is the identity element of the group G. If, on the other hand, $[g, x(n)] = e$ for every $x \in G$ and the corresponding n, then the element g is called a **right Engel element** or a **generalized central element** of the group G.

In what follows we adhere to the terminology of the survey article ([1]), which also contains most of the abstract group-theoretic results needed by us.

The basic information on PI-algebras needed by us is contained in ([2]), Chap. 10. A further development of the results on PI-algebras presented in ([2]) is contained in the works of I. Kaplansky and S. Amitsur cited in ([2]).

It is known (I. Kaplansky, S. Amitsur) that for PI-algebras over an arbitrary commutative ring the basic problems of Bernside type are solved positively. In connection with this, B. I. Plotkin drew the author' s attention to the following problem: will the set of nil-elements in a group embedded in the multiplicative group of a PI-algebra coincide with the locally nilpotent radical; in particular, will the nil group of a PI-algebra be locally nilpotent? The article gives a positive answer to this question, and also shows that the set of generalized central elements of a subgroup of a PI-algebra coincides with the locally nilpotent core. Thus in the present case problems 14.2.7 and 15.3.3 from ([1]) are solved positively. It is also established that the restricted Burnside problem in the case under consideration is solved positively. At the same time, recent results of E. S. Golod ([3]) show that in the general case the most important problems of Burnside type for groups and algebras, including those indicated above, are solved negatively.

Since every matrix group over a commutative ring is embeddable in a PI-algebra, the above results are valid in this case. The results obtained are then applied to the study of nil-elements in topological groups. In a natural way the notion of a topological nil-element is introduced, and the structure of bicompact nilgroups in the topological sense is established. In conclusion it is noted that for topological groups there exists a radical generalizing the locally nilpotent radical in the abstract case and admitting a meaningful theory.

§ 1. The following result was obtained independently and by different methods by the author and B. I. Plotkin.

Theorem 1. *In a matrix group over an arbitrary field, the set of nil-elements coincides with the locally nilpotent radical.*

Proof. Let $N(G)$ and $R(G)$ be, respectively, the set of nil-elements and the radical of the matrix group G. For arbitrary $g_1, g_2 \in N(G)$ and $h \in G$, consider the subgroup $H = \{g_1, g_2, h\}$. By the known

theorem of A. I. Mal' cev (^{4}), the group $H = \lim_{\leftarrow} H_i$, where the H_i are finite matrix groups of one and the same degree. Denote by $g_1^{(i)}, g_2^{(i)}, h^{(i)}$ respectively the images of g_1, g_2, h in H_i. Since in finite groups $N(G) = R(G)$, the elements $g_1^{(i)}$ and $g_2^{(i)}$ belong to the nilpotent normal divisor H_i^0 of the group H_i, whence the solvability of the groups H_i follows. Then, in view of the boundedness of the degree of the matrices from H_i for all i, from the well-known theorem of Zassenhaus and the representability $H = \lim_{\leftarrow} H_i$ the solvability of the group H easily follows. But for solvable groups the theorem is true (see (^{1})); consequently, $g_1 \cdot g_2$ is a nil-element. The local nilpotency of $N(G)$ (see (^{5})) follows from the solvability of $N(G)$ and from the fact that a solvable nil-group is locally nilpotent.

Theorem 2. *In a matrix group Γ over a field, the set of generalized-central elements $O(\Gamma)$ coincides with the locally nilpotent core $I(\Gamma)$.*

Proof. Let $\Delta = \{g_1, g_2, h\}$; $g_1, g_2 \in O(\Gamma)$, $h \in \Gamma$. As in Theorem 1, it is proved that Δ is a solvable group. It is also not hard to show that Δ satisfies the maximality condition. But for groups with the maximality condition the validity of the theorem was established by R. Baer (^{6}). Hence it follows that $O(\Gamma)$ coincides with $I(\Gamma)$.

Theorem 3. *Let G be a subgroup of some PI-algebra F over a commutative ring with identity. Then the set of nil-elements $N(G)$ coincides with the locally nilpotent radical $R(G)$.*

Proof. It is obvious that $N(G) \supseteq R(G)$. Let F^* be the multiplicative group of the algebra F, and let P be the nilradical of F. By a theorem of I. Kaplansky, P is the locally nilpotent radical of F. It is not hard to show that the set $M = \{e + x\}_{x \in P}$ is a locally nilpotent invariant subgroup of F^*. Consider the factor algebra $F = F/P$, which no longer contains nontrivial nil-ideals and, by a theorem of S. Amitsur (^{2}), is embeddable in a semisimple PI-algebra Σ. In turn, the algebra Σ is a subdirect sum of matrix algebras over fields, the degrees of which are bounded in the aggregate. It is easy to see that $F^* = F^*/M \subset \Sigma^*$. Let $M_G = G \cap M$; then $G^* = G/M_G$ is naturally embedded in Σ^*, and $M_G \subset R(G)$. Since Σ^* is a subdirect product of matrix groups of bounded degree, the subgroup Σ^*, which is a subdirect product of solvable groups, is solvable.

Hence, and from Theorem 1, it follows that for $g_1, g_2 \in N(G)$ and $h \in G$ the group $H^* = \{g_1, g_2, h, M_G\}/M_G$ is solvable. Since in a radical group the set of nil-elements forms a subgroup, $g_1 \cdot g_2$ is a nil-element. The local nilpotency of $N(G)$ follows from the local nilpotency of the radical nilgroup.

The following is proved in a completely analogous way.

Theorem 4. *In a subgroup of a PI-algebra the set of generalized-central elements coincides with the locally nilpotent core.*

Corollary. *In a matrix group over an arbitrary commutative ring, the set of nil-elements (respectively generalized-central elements) coincides with the locally nilpotent radical (respectively the core).*

Theorem 5. *A periodic subgroup of a PI-algebra over a commutative ring, the orders of whose elements are bounded in the aggregate, is locally finite.*

Proof. Since every locally nilpotent periodic group is locally finite (see ([7])), by virtue of the reduction carried out in the proof of Theorem 3 it is enough to consider a periodic subgroup $\Phi \subset \Sigma^*$, the orders of whose elements are bounded in the aggregate. Then $\Phi = \prod_s \Phi_\alpha$, where $\prod_s$ denotes the sign of the subdirect product. From Burnside' s theorem (see ([8])) it follows quite easily that every group Φ_α has such a nilpotent normal divisor H_α that

$[\Phi_\alpha : H_\alpha] < e$ for every α. The group $H = \prod_s H_\beta$ is locally finite, hence soluble. Since an extension of a locally finite group by a locally finite group is again locally finite, it suffices to prove the local finiteness of the group $\Phi^* = \prod \Phi_\alpha / H_\alpha$. The latter assertion follows from the known result on the local finiteness of a subdirect product of finite groups whose orders are bounded in the aggregate.

The question of whether the condition of boundedness of the orders of the elements in Theorem 5 is essential remains open.

§ 2. Radicals play a very significant role in abstract group theory. In terms of effectiveness, the most important is the locally nilpotent radical. At the same time, for topological groups not a single topological radical is known, i.e. a radical in the abstract sense which is a closed subgroup, to say nothing of a topological theory of radicals.

Simple examples show that the abstract locally nilpotent radical is in general unsuitable for the study of topological properties, since it is not always closed. Thus, for example, let Γ_α, $\alpha \in \Delta$, be nilpotent bicompact groups whose topological direct product $\Gamma = \prod \Gamma_\alpha$ is not a locally nilpotent group. Then Γ is a bicompact group with an everywhere dense locally nilpotent radical. Nevertheless, for some classes of topological groups the locally nilpotent radical turns out to be closed and coincides with the set of all nil-elements.

In what follows only locally bicompact groups are considered.

Lemma 1. *In a linear group* Li *G, the locally nilpotent radical is closed and is a nilpotent group.*

Proof. Let R be the locally nilpotent radical of G. By a known theorem, R then has a nilpotent normal divisor H of finite index. Since the product of a finite number of nilpotent normal divisors is again a nilpotent normal divisor, R has a characteristic nilpotent subgroup F of finite index. $\bar{F}$ is a nilpotent group, and moreover $\bar{F} \subset \bar{R}$, whence the closedness of R follows. The nilpotency of $\bar{R}$ can be derived, for example, from the fact that the irreducible parts of R are nilpotent, since they possess a center with periodic factor group.

Lemma 2. *The locally nilpotent radical of a connected locally bicompact group G is a closed nilpotent group.*

Proof. First of all we note that Lemma 1 implies the validity of Lemma 2 for Lie groups. Let now B be such a bicompact normal divisor of the group G that $G^* = G/B$ is a Lie group. By Theorem 2 from [9], $B = Z \cdot S$, where Z is a central subgroup in G, and S is connected semisimple. If R is the radical of G, then $Z \subset R$. Consider the group $\widetilde{G} = G/Z$. Then, as is not difficult to see, $\widetilde{R} = R/Z$ is the radical in the group $\widetilde{G} = Z(\widetilde{B}) \cdot \widetilde{B}$, where $Z(\widetilde{B})$ is a Lie group and $Z(\widetilde{B}) \cap \widetilde{B} = (e)$. Since $\widetilde{B}$ has no center, $\widetilde{R} \cap \widetilde{B} = (e)$. If $\tilde{r} = \tilde{f} \cdot \tilde{b}$, where $\tilde{r} \in \widetilde{R}$, $\tilde{f} \in Z(\widetilde{B})$, and $\tilde{b} \in \widetilde{B}$, then for $\tilde{b}_1 \in \widetilde{B}$ we have $\tilde{b}_1 \tilde{r} \tilde{b}_1^{-1} = \tilde{r}_1 = \tilde{f} \cdot \tilde{b}_2$. Then $\tilde{b} = \tilde{b}_2 = \tilde{b}_1 \tilde{b} \tilde{b}_1^{-1}$ for every element $\tilde{b}_1 \in \widetilde{B}$, which entails $\tilde{b} = \tilde{e}$. Thus, $\widetilde{R}$ will be a radical in $Z(\widetilde{B})$, which is a Lie group. Consequently, $\widetilde{R}$ is a closed nilpotent group. Since Z is a central normal divisor of the group G, the analogous assertion is true for R.

Theorem 6. *In a topological group G, whose connected component G_0 is open, the locally nilpotent radical is closed.*

Proof. Let $R_0 = G_0 \cap R$; then from Lemma 2 it follows that R_0 is a closed nilpotent subgroup.

Consider the group $G^* = G/R_0$. Since G_0 is open in G, and $R/R_0 \cap G_0/R_0 = R_0$, the group R/R_0 is discrete, and consequently closed in G^*, which proves the theorem.

Lemma 3. In a Lie group with a finite number of connected components, the set of nil-elements coincides with the nilpotent radical.

The proof is obtained by a natural reduction to matrix groups.

Theorem 7. In a topological group with a finite number of connected components, the set of nil-elements coincides with the nilpotent radical.

The proof of Theorem 7, as well as of Lemma 2, by means of a known result of H. Yamabe (see ([10])), is reduced to the case of Lie groups considered in Lemma 3.

§ 3. **Definition.** An element g of a topological group G is called a **nil-element in the topological sense** if, for every $x \in G$,

$$\lim_{n \to \infty} [x, g(n)] = e$$

in the topology of the group G. It is clear that every nil-element in the abstract sense is topological, and in any topology.

A topological group Γ is called a **topological nil-group** if all its elements are topological nil-elements. Recall, following ([12]), that a topological group is called **locally projectively nilpotent** if every finitely generated subgroup of it is projectively nilpotent.

The following topological generalizations of the nil-problems in the abstract case arise naturally: 1) When will a topological nil-group be locally projectively nilpotent? 2) When does the set of topological nil-elements form a subgroup?

From Theorem 7, § 2, and Theorem 6 of ([11]), the following theorem follows:

Theorem 8. A bicompact topological nil-group is projectively nilpotent.

As for Problem 2, it is answered negatively already in the simplest cases. There exist, for example, compact Lie groups, and also connected solvable Lie groups, in which the degree of a topological nil-element is no longer a nil-element.

In conclusion we note without proof the following result:

Theorem 9. Every locally bicompact group possesses a locally projectively nilpotent radical.

Thus, Theorem 9 shows that for locally bicompact groups a topological theory of the radical is possible.

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Note: Figure translations are in progress. See original paper for figures.