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Abstract

Full Text

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MATHEMATICS

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UNDECIDABILITY IN CONSTRUCTIVE PREDICATE CALCULUS OF CERTAIN CLASSES OF FORMULAS CONTAINING ONLY UNARY PREDICATE VARIABLES

(Presented by Academician P. S. Novikov, 9 I 1965)

It is known that, in the class of formulas containing only unary predicate variables, the derivability problem is algorithmically decidable in the classical predicate calculus without function symbols. The main purpose of the present communication is to prove, by constructive means, the undecidability of the derivability problem in the constructive, minimal, and positive predicate calculi without function symbols* for the class of formulas containing only one unary predicate variable. In the present communication the terminology and results of paper (1) are used.

1. Let P and P' be arbitrary unary predicate variables, and R an arbitrary binary predicate variable. Then Theorems 1–5 hold.

Theorem 1. The class of absolute sequents containing the predicate variable R, and containing no other predicate variables and no positive occurrences of the sign $\vee$, is a decision class for C, K, M, and A.

Theorem 2. The class of sequents containing the predicate variable P, and containing no other predicate variables, no sign $\neg$, and no positive occurrences of the quantifier $\forall$, is a decision class for K, M, and A.

Theorem 3. The class of sequents containing the predicate variable P, and containing no other predicate variables, no occurrences of the sign $\supset$, and no positive occurrences of the sign $\vee$ and of the quantifier $\forall$, is a decision class for K and M.

Theorem 4. The class of positive sequents containing the predicate variables P and P', and containing no other predicate variables, is a decision class for K, M, and A.

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Theorem 5. The class of quasipositive sequents containing the predicate variable P, and containing no other predicate variables, is a decision class for K and M.

2. We outline the proofs of these theorems. In paper (1) it is shown that the class of absolute sequents of the form

$$Q\mathcal{M} \to \exists x_1 \dots x_n (P_1(x_1, \dots, x_n) \& P_2(x_1, \dots, x_n)), \tag{1}$$

where Q is a prefix and $\mathcal{M}$ is a disjunction of conjunctions of elementary formulas containing no predicate variables distinct from P_1 and P_2, is a decision class for C, K, M, and A.

* These calculi will be denoted respectively by K, M, and A, and the classical predicate calculus without function symbols by C.

We shall construct algorithms $\varphi_1, \varphi_2, \varphi_3, \varphi_4$, and φ_5 such that, whatever the absolute sequent S of the form (1):

1) the algorithm φ_1 transforms S into an absolute sequent containing the predicate variable R and containing no positive occurrences of the sign $\vee$;
2) the algorithm φ_2 transforms S into a sequent containing the predicate variable P and containing no occurrences of the sign $\neg$ and no positive occurrences of the quantifier $\forall$;
3) the algorithm φ_3 transforms S into a sequent containing the variable P and containing no occurrences of $\supset$ and no positive occurrences of $\vee$ or of the quantifier $\forall$;
4) the algorithm φ_4 transforms S into a positive sequent containing the predicate variables P and P';
5) the algorithm φ_5 transforms S into a quasi-positive sequent containing the predicate variable P;
6) the sequent S is derivable in **C** if and only if the sequent $\varphi_1(S)$ is derivable in **C**;
7) whatever i may be ($1 \leqslant i \leqslant 5$), the sequent S is derivable in **K** if and only if the sequent $\varphi_i(S)$ is derivable in **K**;
8) whatever i may be ($1 \leqslant i \leqslant 5$), the sequent S is derivable in **M** if and only if the sequent $\varphi_i(S)$ is derivable in **M**;
9) whatever i may be ($i = 1, 4$), the sequent S is derivable in **A** if and only if the sequent $\varphi_i(S)$ is derivable in **A**.

Let S be a sequent of the form (1). The result of applying the algorithm φ_1 to the sequent S shall be taken (by definition) to be the sequent obtained from S by replacing each occurrence of an elementary formula of the form $P_k(y_1, \dots, y_n)$ $(k = 1, 2)$ by the formula

$$\exists \alpha_1 \dots \alpha_{n+k+1} \left(\mathop{\&}_{i=1}^{n+k} R(\alpha_i, \alpha_{i+1}) \;\&\; R(\alpha_{n+k+1}, \alpha_1) \;\&\; \mathop{\&}_{i=1}^{n} (R(\alpha_i, y_i) \;\&\; R(\alpha_{i+1}, y_i)) \right), \tag{2}$$

where $\alpha_1, \dots, \alpha_{n+k+1}$ are pairwise distinct individual variables.

The result of applying the algorithm φ_2 to the sequent S shall be taken (by definition) to be the sequent obtained from $\varphi_1(S)$ as a result of replacing each occurrence of a subformula of the form $R(x, y)$ by the formula

$$\exists \alpha\beta \left((P(y) \supset P(x)) \supset (P(\alpha) \vee P(\beta))\right). \tag{3}$$

The result of applying the algorithm φ_3 to the sequent S shall be taken to be the sequent obtained from $\varphi_1(S)$ as a result of replacing each occurrence of an elementary formula of the form $R(x, y)$ by the formula

$$\exists \alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3 \left(\mathop{\&}_{i=1}^{3} \neg \left(P(\beta_i) \;\&\; \mathop{\&}_{j=1}^{3} P(\alpha_j) \right) \right) \;\&$$

$$\&\neg \left(P(\alpha_1) \;\&\; P(x) \;\&\; \mathop{\&}_{i=1}^{3} P(\beta_i) \right) \;\&\neg \left(P(\alpha_2) \;\&\; P(\beta_1) \;\&\; P(\beta_2) \;\&\; P(x) \;\&\; P(y)\right). \tag{4}$$

The result of applying the algorithm φ_4 to S shall be taken to be the result of replacing in $\varphi_1(S)$ each occurrence of a subformula of the form $R(x, y)$ by the formula

$$\exists \alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3 \; \forall \gamma \, (P'(\gamma) \vee \mathfrak{A}), \tag{5}$$

where by $\mathfrak{A}$ is denoted the formula

$$\left(\mathop{\&}_{i=1}^{3} \left(P(\beta_i) \vee \bigvee_{j=1}^{3} P(\alpha_j) \right) \right) \;\&\; \left(P(\alpha_1) \vee P(x) \vee \bigvee_{j=1}^{3} P(\beta_j) \right) \;\&$$

$$\& \left(P(\alpha_2) \vee P(\beta_1) \vee P(\beta_2) \vee P(x) \vee P(y)\right).$$

The result of applying the algorithm φ_5 to S will be taken to be the result of replacing in $\varphi_1(S)$ each occurrence of a subformula of the form $R(x, y)$ by the formula

$$\exists \alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3 \forall \gamma (\neg\neg P(\gamma) \vee \mathfrak{A}). \tag{6}$$

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Since a sequent S is derivable in **K** if and only if it is derivable in **C**, **M**, and **A***, in order to complete the proof of Theorems 1-5 it remains only to show that from the derivability of the sequent $\varphi_i(S)$ $(1 \leq i \leq 5)$ in **K** there follows the derivability of S in this calculus. To prove the latter assertion it is enough to note that, for each i $(i = 1, \dots, 5)$, a derivation of the sequent $\varphi_i(S)$ in the constructive calculus G_2 (see ([2])) can, using the results of ([3]), be transformed into a derivation of the same sequent in which all formulas in the axioms have the form $(i + 1)$.

3. If B is a sequent, then by $\{B\}$ we shall denote the formula image of the sequent B. Let P be an arbitrary unary predicate variable. Then the following Theorems 6 and 7 hold:

Theorem 6. The class of formulas of the form $(\{B\} \vee \neg\{B\})$, where B is a quasi-positive sequent containing the predicate variable P and containing no other predicate variables, is a reduction class for **K** and **M**.

Theorem 7. The class of formulas of the form $(\neg\neg\{B\} \supset \{B\})$, where B is a quasi-positive sequent containing the predicate variable P and containing no other predicate variables, is a reduction class for **K** and **M**.

Corollary 1. There is no algorithm recognizing the formulas derivable in **K** among the formulas derivable in **C**.

Corollary 2. There is no algorithm recognizing the formulas derivable in **K** among the formulas whose double negation is derivable in **K**.

Theorems 6 and 7 follow from the following lemma.

Lemma. Whatever the sequent S of the form (1) may be, the formulas $\neg\{\varphi_2(S)\}$, $\neg\{\varphi_3(S)\}$, $\neg\{\varphi_4(S)\}$, and $\neg\{\varphi_5(S)\}$ are derivable in the constructive predicate calculus.

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* See Theorem 1 of the paper ([1]).

Note: Figure translations are in progress. See original paper for figures.

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