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E. A. Loginov

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Abstract

Full Text

E. A. Loginov

On Proper Three-Coloring

(Presented by Academician L. S. Pontryagin, January 7, 1965)

The present article concerns proper coloring of the vertices of a graph without loops (a loop is an edge closed on itself). A **proper** coloring of the vertices of a graph (of the regions of a map) is one in which any two mutually adjacent vertices (regions) have different colors. A graph is said to be p -chromatic if its vertices cannot be properly colored with $(p - 1)$ colors, but can be properly colored with p distinct colors. It is proved here that the chromatic number of any connected graph without loops whose vertices have degree ≤ 3 , with the sole exception of the tetrahedral graph, is at most three (the degree of a vertex is the number of edges issuing from that vertex). Passing to the dual map, the result of the article may be expressed as follows: with the sole exception of the tetrahedral map, any map consisting solely of triangular regions on any surface admits a proper three-coloring.

Theorem. *If a connected non-tetrahedral graph without loops contains vertices of degree ≤ 3 , then its chromatic number is ≤ 3 .*

We first consider auxiliary definitions and properties. A **component** $\alpha - \beta$ of a graph Γ with properly colored vertices is a connected subgraph Γ' containing only vertices of colors α, β , together with the edges joining them, such that every vertex of the graph Γ not belonging to the subgraph Γ' but adjacent to one of its vertices is colored in a color different from α and β . From the definition the following properties of a component follow:

Property 1. Interchanging the colors α, β of all vertices of the component $\alpha - \beta$ does not affect the propriety of the coloring of the vertices of the graph Γ .

Property 2. If two vertices of colors 1) α and α , or 2) β and α , do not belong to one component $\alpha - \beta$, then, without violating the propriety of the vertex coloring, one may pass from case 1) to case 2), and conversely. Indeed, for this it suffices, in the component $\alpha - \beta$ passing through the first of the indicated vertices, to interchange α and β everywhere.

Lemma 1. *If in a component $\alpha - \beta$ of a three-chromatic graph with vertices of degree ≤ 3 there are simultaneously end vertices A and B and at least one cycle, then it is always possible, without violating the propriety of the coloring, to arrange that A and B will belong to two different components $\alpha - \beta$.*

An **end vertex** C of a component $\alpha - \beta$ is one such that not more than one vertex adjacent to C belongs to the component $\alpha - \beta$.

Proof of Lemma 1. Consider only the component $\alpha - \beta$, discarding the remaining elements of the graph Γ . On a path from the end vertex A to B there is always a vertex all three of whose neighboring vertices belong to the component $\alpha - \beta$. Giving the first such vertex on the path from A to B the color γ , we split the component $\alpha - \beta$ into several others, different ones of which contain A and B .

Lemma 2. *If in a component $\alpha - \beta$ of a three-chromatic graph whose vertices are of degree not higher than three there are end vertices A , B , and C , then, without violating the propriety of the coloring, one can arrange that three different components $\alpha - \beta$ pass through A , B , and C .*

Proof. If the component under consideration has a cycle, then Lemma 2 follows from Lemma 1. When there is no cycle, we find such a vertex M of the component $\alpha - \beta$ to which the paths from A to B and from A to C , going ...

along the edges of the component $\alpha - \beta$, coincide, and then diverge. Three vertices adjacent to M belong to the component $\alpha - \beta$, and the vertex M can be assigned the color γ without violating the propriety of the coloring. This gives the property noted in Lemma 2.

Definition. We shall call an edge of a graph a **typical edge** if in the graph there is no vertex adjacent to both endpoints of this edge.

Lemma 3. *Every connected non-tetrahedral graph containing vertices only of degree 3, in which there are no loops and no cycles consisting of exactly 2 edges, contains at least one typical edge.*

Proof. If an arbitrary edge CD is not typical, then one or two vertices are adjacent simultaneously to each of the vertices C and D .

A. Only one vertex M is adjacent to C and D simultaneously. Then, in addition to M , a vertex E is adjacent to C , and F to D ; moreover, E and F cannot both be adjacent to M simultaneously, since there are no vertices of degree greater than 3. Suppose M is not adjacent to E . Then CE is a typical edge.

B. The vertices M and N are both adjacent simultaneously to C and D . Since the graph under consideration is not tetrahedral, the vertex M is not adjacent to the vertex N , and therefore the vertex M , in addition to C and D , has some neighboring vertex L . Then the edge ML is typical: in the graph all vertices have degree no greater than 3, and therefore the two vertices of the edge CD , adjacent to M and N respectively, cannot be adjacent to L .

Proof of the theorem. Consider any connected, loopless, non-tetrahedral graph Γ whose vertices have degree ≤ 3 . If Γ contains a cycle formed by exactly two arbitrary edges, then in proving the theorem it suffices to consider the simplified graph Γ' , obtained by contracting each cycle of 2 edges to a point, so that instead of the cycle there is formed a vertex of degree 2. This does not destroy the connectedness of the graph, no vertices of degree > 3 arise, and a tetrahedral graph is not formed (which contains exactly 4 vertices, each of

Fig. 1

Figure 1: Fig. 1

degree 3). Repeating the operation of contracting a cycle of 2 edges to a point until all such cycles disappear, we obtain a graph Γ' . Clearly, if the graph Γ' is three-chromatic, then the graph Γ is also three-chromatic. If in the graph Γ' there is a vertex of degree < 3 , then let us reduce to zero the length of an edge issuing from this vertex, merging the indicated vertex of degree < 3 with a neighboring one; this does not create loops and leads to the formation, from two vertices, of one vertex of degree ≤ 3 , without destroying the connectedness of the graph. If the graph thereby obtained is three-chromatic, then, clearly, the chromatic number of the original graph is no greater than three. Repeating the indicated operation, one can always pass from the graph Γ' to such a connected non-tetrahedral graph Γ'' , without loops and without vertices of degree > 3 , that either 1) the graph Γ'' contains exactly 5 vertices, at least one of which has degree ≤ 2 , or 2) the graph Γ'' consists of no fewer than 5 vertices, each of degree 3. We note that if, in the course of simplifying the graph Γ' , the intermediate graph had simultaneously cycles of 2 edges and vertices of degree ≤ 2 , then we always first carry out the operation of eliminating the cycle.

Let us consider separately cases 1) and 2).

- 1) If among the 5 vertices there is only one vertex of degree < 3 , then it is easy to show that the given graph is obtained from the tetrahedral one by dividing one of its edges by an additional fifth vertex into 2, and such a graph is three-chromatic, as is easily verified directly. But if in the graph of 5 vertices there is more than one vertex of degree < 3 , then we simplify this graph by the method set out above, reducing the number of vertices in the graph to 3; obviously, the chromatic number of such a graph is ≤ 3 .

Thus, in order to prove that the graph Γ is three-chromatic, it remains to consider case 2), i.e., it remains to prove that a connected nonte-

the cubic graph Γ'' without loops, all of whose vertices have degree 3, is three-chromatic. We shall prove this using Lemma 4.

- 2) We shall call the **discarding** of a typical edge AB its removal from the graph together with the vertices A and B , and the replacement of the edges CA and AE (DB and BF) adjacent to the edge AB at the point A (B) by a single edge CE (DF).

Lemma 4. If the graph Γ''' , which is formed by discarding a typical edge from a connected noncubic graph Γ'' without loops, containing only vertices of degree 3, is three-chromatic, then the graph Γ'' is also three-chromatic.

Fig. 1

Proof of Lemma 4. We discard in the graph Γ'' the typical edge AB , thereby

reducing the number of vertices in it by 2 and not forming loops. The possible cases are 1° and 2°, which we shall consider separately.

1°. Among the colors of the vertices of the graph Γ''' : C, D, E , and F , adjacent in Γ'' to A or B , there are all three colors: α, β, γ (Fig. 1). It is always possible to give A and B colors different from the colors of their adjacent vertices. Thus, to that vertex of the edge AB which is adjacent to already colored vertices of different colors, we give the missing third color. Then the other vertex of the edge AB will have three colored adjacent vertices, two of which are of the same color. As a result, this vertex of the edge AB can be given a color different from the color of its adjacent vertices.

2°. In the graph Γ''' , any two (for example C, D) of the vertices C, D, E , and F have color α , and the other two have color β . For the proof it is convenient to split the edges CE and DF , forming from them respectively the edges CA_1, A_2E and DB_1, B_2F . We give the vertices A_1, A_2, B_1 , and B_2 the color γ , without violating the properness of the coloring of the graph Γ^* thereby obtained from Γ''' .

We shall prove Lemma 4 by contradiction. Suppose (condition w) that, although the graphs Γ''' and Γ^* are three-chromatic, the chromatic number of the graph Γ'' is not less than 4. Then we arrive at the following. The vertices C and E (and also D and F) of the graph Γ^* are connected by an $\alpha - \beta$ component, since otherwise, using property 2, it would be possible to make C and E (D and F) of one color (which means that the graph Γ'' would be three-chromatic). In exactly the same way, C and D (E and F) are connected by an $\alpha - \gamma$ ($\beta - \gamma$) component. Let us consider in more detail the $\alpha - \gamma$ component joining C and D of the graph Γ^* , which we shall call Δ . In Δ there are necessarily exactly 2 terminal vertices (otherwise, by Lemma 2, we arrive at a contradiction with (w)), and these are, obviously, precisely A_1 and B_1 . It follows from Lemma 1 that in Δ there is not a single cycle and every vertex, except A_1 and B_1 , has in Γ^* exactly one adjacent vertex of color β . Take in Γ^* any such vertex M of color β which is adjacent to a vertex J of color γ of the component Δ . The vertices K and N , adjacent to M , necessarily have colors α and γ , since, if they were both of color a) α or b) γ , then we could: in case a), interchange the colors of M and J ; in case b), color M with color α , then give J color β , and, by (w), both are impossible.

Since all vertices C, D, E, F , and M belonging to $\alpha - \beta$ components are terminal, it follows from Lemma 2 that no 3 of the 5 indicated vertices belong to one $\alpha - \beta$ component. Since C and E (D and F) must be connected to each other by an $\alpha - \beta$ component, M cannot be connected by an $\alpha - \beta$ component with any of the vertices C, D, E , or F .

We interchange colors in the $\alpha - \beta$ component passing through M . Such an interchange: a) will not change the colors of C, D, E , and F and will not violate the properness of the coloring of the vertices of the graph Γ^* ; b) will not violate the component Δ and (therefore) will not create cycles in it.

Fig. 2

Figure 2: Fig. 2

Indeed, it is not possible simply to break the component Δ by condition (w); it is impossible to break it in one place and connect it in another, or to form a cycle, since, on the basis of Lemma 2, one may assume that

the vertex M is not connected by the $\alpha - \beta$ component simultaneously with two end vertices of color α .

Consequently, after the interchange, the vertex J is surrounded by three vertices of color α , and, by assigning J color β , we break Δ . Then, interchanging α and γ in the $\alpha - \gamma$ component passing through C , we obtain color γ at C , and the former colors at D, E , and F . But then the arguments of case 1° are applicable, which gives a result contradicting (w). Lemma 4 is proved.

Fig. 2

When a typical edge is discarded from Γ'' , a graph Γ''' is formed (with the number of vertices smaller by 2), either connected or not. In the latter case, in order that every connected subgraph $B'(B'')$ obtained from the graph Γ'' should not be tetradical, we agree that, when two disconnected subgraphs of the graph Γ'' are formed by discarding a typical edge AB , the vertices A and B themselves are to be left. Let us apply the method of induction to the proof of the theorem. We see that a graph Γ satisfying the condition of the theorem will be three-chromatic whenever the graph Γ'' , satisfying the condition of the theorem and having the same or a smaller number of vertices, is three-chromatic. It is shown further that the graph Γ'' either contains 5 vertices and is therefore three-chromatic, or else satisfies the condition of the theorem and contains more than 4 vertices, each of which has degree 3. In the latter case, with the help of Lemma 4, the graph Γ'' is reduced to the graph Γ''' or to 2—taking into account the remark made—to graphs B' and B'' , satisfying the condition of the theorem and having a smaller number of vertices. To the graphs Γ''' , B' , and B'' all the arguments which applied to the original graph Γ are applicable. In the end one can always reduce the graph Γ to one or several connected graphs without loops and without vertices of degree > 3 , but with the number of vertices equal to 5 (the graph B_5) or 6 (the graph B_6). If among the vertices of the graph B_6 there is at least one vertex of degree less than 3, then we simplify it in the indicated way, reducing the number of vertices to 5 and obtaining the graph B_5 . If the graph contains 5 vertices of degree ≤ 3 , then at least one of them has degree < 3 . Earlier (see case 1) it was shown that in this case the graph B_5 is three-chromatic.

Let now all 6 vertices of B_6 have degree 3. Then, by Lemma 3, it always contains either a cycle of 2 edges or a typical edge AB . In the first case, by contracting the cycle to a point, the graph B_6 is reduced to the three-chromatic graph B_5 already considered; in the second case, discarding AB either leads

to the formation from B_6 of a nontetradical graph with 4 vertices, and then, with the help of the arguments used above, we finally reduce it to a graph with number of vertices (and hence colors) not exceeding 3, or it gives a tetradical graph, and then the graph B_6 (Fig. 2) is two-chromatic. Thus, the chromatic number of the graphs B_5 and B_6 is not greater than 3, and, consequently, it has been proved that the chromatic number of the intermediate graphs B' , B'' , Γ' , ..., and of the original graph Γ is not greater than 3.

If on an arbitrary surface one takes any map made of triangles and assigns to it its dual graph, then we arrive at the conclusion that on an arbitrary surface any map made of triangles, except a tetradical one, admits a proper three-coloring.

The theorem proved here is a special case of the following general theorem, whose proof is also not difficult:

The chromatic number of any graph with vertices of degree $\leq n$ does not exceed n ($n > 2$), except for the case of a graph consisting only of the one-dimensional skeleton of the n -simplex.

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Note: Figure translations are in progress. See original paper for figures.

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