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Abstract

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TENSOR PRODUCTS AND FUNCTORS IN CATEGORIES OF BANACH SPACES DEFINED BY KB-LINEALS

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In the author' s note ([1]), for each KB-lineal E a functor Φ_E was constructed in the category of Banach spaces, which is a broad and natural generalization of the functors defined by minimal normed ideals of sequences and measurable functions ([2]). The functor Φ_E assigns to a Banach space X the completion of the algebraic tensor product $E \otimes X$ with respect to the crossnorm n_E, defined by the formula

$$n_E\left(\sum_{k=1}^{n} e_k \otimes x^k\right) =$$

$$= \inf\left\{\|u\|_E : \ u \geqslant \left|\sum_{k=1}^{n} e_k \langle x_k, x'\rangle\right| \text{ for all } x' \in X', \ \|x'\| \leqslant 1\right\};$$

this completion will henceforth be denoted by $E\widetilde{\otimes}X$. In ([1]) the properties of the functor Φ_E were studied; in particular, the space of mappings $\{\Phi_E \to \Phi_F\}$ of one such functor into another and the dual functor $D\Phi_E$ were described.

In the present note, which is a continuation of ([1]), further properties of the tensor product $E\widetilde{\otimes}X$ and of the functor Φ_E are reported. For definitions relating to functors in categories of Banach spaces, see ([2]).

An L-space is called, following S. Kakutani ([3]), a KB-lineal E satisfying the condition $\|e_1+e_2\| = \|e_1\|+\|e_2\|$ for any positive $e_1, e_2 \in E$; in ([3]) it is proved that every L-space is linearly isometric and structurally isomorphic to the KB-lineal L^1_M, where $M = (S, \Sigma, \mu)$ is some measure space.

An M-space is called, following S. Kakutani ([4]), a KB-lineal E satisfying the condition $\|\sup(e_1, e_2)\| = \max(\|e_1\|, \|e_2\|)$ for any positive $e_1, e_2 \in E$; in ([4]) it is proved that every M-space is linearly isometric and structurally isomorphic to a sublineal of a special type of the KB-lineal C_Q, where Q is some bicompactum.

Theorem 1. Let E be a KB-lineal.

1) In order that, for every X, the crossnorm n_E coincide on $E \otimes X$ with the maximal crossnorm π, it is necessary and sufficient that E be an L-space.
2) In order that, for every X, the crossnorm n_E coincide on $E \otimes X$ with the minimal crossnorm ε, it is necessary and sufficient that E be an M-space.

We shall need some definitions from note ([1]); let us recall them. We say that for a KB-lineal E condition $(*)$ is fulfilled if, for every monotonically increasing sequence bounded above of elements $e_n \geqslant 0$ of E, $\|e_n\| \to \infty$ as $n \to \infty$; condition $(**)$, if for every monotonically increasing sequence of elements $e_n \geqslant 0$ of E having a supremum $e = \sup_n e_n$, $\|e_n\| \to \|e\|$ as $n \to \infty$; condition $(***)$, if for every Banach space X and every

for $z \in E \otimes X$

$$n_E(z) = \inf \left\| \sum_{k=1}^{n} |e_k| \, \|x_k\| \right\|_E ,$$

where the lower bound is taken over all possible representations

$$z = \sum_{k=1}^{n} e_k \otimes x_k .$$

Condition $(*)$ is satisfied by the majority of known KB-lineals, in particular by all KB-spaces ([5,6]); it is not satisfied by the KB-lineal C_0. Conditions $(**)$ and $(***)$ are fulfilled in all known cases.

We call a mapping $\alpha : E \to Y$ of a KB-lineal E into a Banach space Y **summing** if it carries every series $\sum_{k=1}^{\infty} e_k$ from E for which the series $\sum_{k=1}^{\infty} |e_k|$ converges into an absolutely convergent series $\sum_{k=1}^{\infty} \alpha e_k$ in Y.

We call a mapping $\alpha : X \to E$ of a Banach space X into a KB-lineal E **proper** if it carries the unit ball of X into a subset of E that is bounded in the sense of the ordering.

Theorem 2. Let E be a KB-lineal satisfying condition $(***)$, and let X be a Banach space. Then the space $(E \widetilde{\otimes} X)'$, conjugate to $E \oplus X$, is isometric to the space of summing mappings of E into X', endowed with the norm

$$\nu_E(\alpha) = \sup \left[\sum_{k=1}^{n} \|\alpha e_k\|_{X'} \Big/ \left\| \sum_{k=1}^{n} |e_k| \right\|_E \right] ,$$

where the least upper bound is taken over all possible finite collections of elements $e_1, \dots, e_n \in E$.

The space E', conjugate to the KB-lineal E, is a conditionally complete KB-lineal with respect to the natural order (see ([5]) or ([6])).

Proposition 1. Let E be a KB-lineal, X a Banach space, and let $\alpha : X \to E$ be a proper mapping. Then its conjugate $\alpha^* : E' \to X'$ is a summing mapping and

$$\nu_{E'}(\alpha^*) \le n(\alpha) = \inf\{\|u\|_E : \ u \ge |\alpha x| \text{ for all } x \in X, \ \|x\| \le 1\}.$$

Proposition 2. Let E be a KB-lineal, Y a Banach space, and X its closed subspace. Then $E \widetilde{\otimes} X$ is a closed subspace of $E \widetilde{\otimes} Y$.

Proposition 3. Let E be a KB-lineal satisfying condition $(***)$, X a Banach space, X_0 its closed subspace, and $Y = X/X_0$. Then $E \widetilde{\otimes} Y$ is isometric to the quotient space

$$E \widetilde{\otimes} X \,/\, E \widetilde{\otimes} X_0.$$

Let E be a KB-lineal and X a Banach space. The identity mapping of $E \widetilde{\otimes} X$ onto itself extends to a continuous linear mapping

$$\omega : \Phi_E(X) \equiv \widetilde{E \otimes X} \to E \widehat{\otimes} X^*.$$

We say that the functor Φ_E satisfies the **condition of one-to-one correspondence** if the mapping ω is one-to-one for every X in the category of Banach spaces under consideration.

Theorem 3. Let the KB-lineal E satisfy condition $(***)$, and let Φ_E satisfy the condition of one-to-one correspondence. Then the functor Φ_E is exact, i.e., every exact triple

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$$

is carried by it into the exact triple

$$\Phi_E(X) \xrightarrow{\Phi_E(\alpha)} \Phi_E(Y) \xrightarrow{\Phi_E(\beta)} \Phi_E(Z).$$

In ([1]) a realization of the space $\{\Phi_E \to \Phi_F\}$ was described in the form of the space $(E \to F)_r$ of regular mappings of E into F. The space $\{\Phi_E \to \Phi_F\}$ admits a somewhat different description under other conditions on the KB-lineals E and F.

* $\widetilde{E \otimes X}$ denotes, as usual, the completion of $E \otimes X$ with respect to the minimal crossnorm.

Denote by c_0^n the n-dimensional coordinate space of elements $x = (\xi_1, \dots, \xi_n)$ with norm $\|x\|_{c_0^n} = \max_{1 \le k \le n} |\xi_k|$. Let E, F be KB-linear spaces. A mapping $\alpha \in (E \to F)$ will be called K-bounded if it sends every sequence (e_k) bounded in the sense of the ordering in E into a sequence (αe_k) of the same kind in F.

We shall say that a KB-linear space E has property (N) if, for every monotonically increasing and bounded above sequence of elements $e_n \ge 0$ $(n = 1, 2, \dots)$ from E and every number $\delta > 0$, there exists an element $e_\delta \in E$ majorizing all the e_n and such that

$$\|e_\delta\| \le \lim_{n \to \infty} \|e_n\| + \delta.$$

For a conditionally σ-complete KB-linear space, conditions (N) and () **are equivalent; in the general case (N) implies** ().

Theorem 4. *Let E, F be KB-linear spaces satisfying conditions ()* and (N); let $\mathcal{K}$ be a category of Banach spaces containing spaces X_n for which there exist isometric embeddings $i_n : c_0^n \to X_n$ $(n = 1, 2, \dots)$. Then, for the functors Φ_E and Φ_F considered in $\mathcal{K}$, the space $\{\Phi_E \to \Phi_F\}$ is isometric to the space of K-bounded mappings $\alpha : E \to F$, endowed with the norm*

$$\|\alpha\|_0 = \sup \left[\left\| \sup_{1 \le k \le n} |\alpha e_k| \right\|_F \Big/ \left\| \sup_{1 \le k \le n} |e_k| \right\|_E \right],$$

where the supremum is taken over all finite sets of elements $e_1, \dots, e_n \in E$.

Remark. Theorem 3 in ([1]) is formulated with a superfluous restriction on the category $\mathcal{K}$; for its validity it is enough to impose on $\mathcal{K}$ the same condition as in Theorem 4 of the present paper. Similarly, Theorem 4 in ([1]) holds if one requires only that $\mathcal{K}$ contain spaces X_n for which there exist isometric embeddings $i_n : l_n^1 \to X_n$, where l_n^1 is the n-dimensional coordinate space of elements

$$x = (\xi_1, \dots, \xi_n) \quad \text{with norm} \quad \|x\|_{l_n^1} = \sum_{k=1}^{n} |\xi_k|.$$

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Note: Figure translations are in progress. See original paper for figures.