ON COMPLETE REGULARITY OF MULTIDIMENSIONAL STATIONARY PROCESSES WITH DISCRETE TIME

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Abstract

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ON COMPLETE REGULARITY OF MULTIDIMENSIONAL STATIONARY PROCESSES WITH DISCRETE TIME

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1. Let $x(t)=(x_1(t),\dots,x_n(t))$ be an n-dimensional stationary process in the broad sense with discrete time $t=\dots,-1,0,1,\dots$. We shall assume $Ex(t)=0$; by $\mathbf{f}(\lambda)=\|f_{ij}(\lambda)\|$, $i,j=1,\dots,n$, we denote the spectral (matrix) density (s.d.) of the process $x(t)$ (see (1))*. Denote by H_k^l, $k\le l$, the closed linear span, in mean square, of the quantities $x_j(t)$, $k\le t\le l$, $j=1,\dots,n$.

We shall say that the process $x(t)$ is completely regular if, for all $\xi\in H_{-\infty}^0$, $\eta\in H_i^{\infty}$, $\tau\ge 0$,

$$|E\xi\bar\eta|\le\alpha(\tau)(E|\xi|^2)^{1/2}(E|\eta|^2)^{1/2}, \tag{1}$$

where $\alpha(\tau)\downarrow 0,\ \tau\to\infty$.

As follows from results of A. N. Kolmogorov and Yu. A. Rozanov (2), for Gaussian processes $x(t)$ condition (1) is equivalent to the strong-mixing condition of M. Rosenblatt (3). In the case of arbitrary stationary processes in the broad sense, the term complete regularity (it was used by Yu. A. Rozanov (1)) seems more natural, since there is a deep connection between condition (1) and the condition of regularity of stationary processes in the sense of A. N. Kolmogorov (see, for example, (4)).

In the present note we study what conditions should be imposed on the s.d. $\mathbf{f}(\lambda)$ in order to ensure a particular rate of decrease of the mixing coefficient (cf. (5)). Since every completely regular process is regular, its s.d. $\mathbf{f}(\lambda)$ has constant rank $m\le n$ for almost all $\lambda\in[-\pi,\pi]$ (1). In § 2, processes of full rank $m=n$ are considered; in § 3, degenerate processes, $m<n$.

The proofs of all the results given below are based on the following analytic reformulation of (1):

$$\alpha(\tau)=\alpha(\tau;\mathbf{f})=\sup_{\vec{\varphi},\vec{\psi}}\left|\int_{-\pi}^{\pi}e^{i\tau\lambda}(\mathbf{f}(\lambda)\vec{\varphi}(\lambda),\vec{\psi}(\lambda))\,d\lambda\right|, \tag{2}$$

where, in general, $(A\vec{\alpha}, \vec{\beta}) = \sum A_{kj}\alpha_k\bar{\beta}_j$; the supremum is taken over all vectors $\vec{\varphi}(\lambda)$, $\vec{\psi}(\lambda)$, all of whose coordinates φ_k, ψ_i are elements of the Hardy space H_1 and

$$\int_{-\pi}^{\pi} (\mathbf{f}(\lambda)\vec{\varphi}(\lambda), \vec{\varphi}(\lambda))\, d\lambda = \int_{-\pi}^{\pi} (\mathbf{f}(\lambda)\vec{\psi}(\lambda), \vec{\psi}(\lambda))\, d\lambda = 1.$$

2. We shall agree to say that a matrix function $\mathbf{f}(\lambda) = \|f_{ij}(\lambda)\|$ is continuous, differentiable, etc., if all its elements $f_{ij}(\lambda)$ are continuous, differentiable, etc.

* Here and below, bold Latin letters denote matrices and matrix functions; Greek letters with arrows denote vectors and vector functions with values in E_n.

Theorem 1. *If the s.d.* $\mathbf{f}(\lambda)$ *is continuous, and* $\det \mathbf{f}(\lambda) > c > 0$*, then the process* $x(t)$ *is completely regular. If, moreover, by* $\bar{E}_s(h)$ *we denote the magnitude of the best approximation of the function* $h(\lambda)$ *by trigonometric polynomials of degree* $\leq s$*, then*

$$\alpha(\tau) = O(\max E_{\tau-1}(f_{ij})). \tag{3}$$

Theorem 2. *In order that* $\alpha(\tau) = O(\tau^{-r-\beta})$*,* $r = 0, 1, \ldots,$ $0 < \beta < 1$*, it is necessary and sufficient that the s.d.* $\mathbf{f}(\lambda)$ *be representable in the form*

$$\mathbf{f}(\lambda) = \mathbf{P}(\lambda)\mathbf{g}(\lambda)\mathbf{P}^*(\lambda), \tag{4}$$

where $\mathbf{P}(\lambda)$ *is a polynomial matrix-function, while the matrix* $\mathbf{g}(\lambda)$ *has an* r*-th derivative satisfying a Hölder condition of order* β*, and* $\det \mathbf{g}(\lambda) > c > 0$*.*

Theorem 3. *In order that* $\alpha(\tau) = O(e^{-\delta\tau})$*,* $\delta > 0$*, it is necessary and sufficient that the s.d.* $\mathbf{f}(\lambda)$ *admit an analytic continuation into the strip of values of the complex argument* $z = \lambda + i\mu$ *of width* 2δ*.*

Theorem 4. *In order that* $\alpha(\tau) = O(e^{-\delta\tau})$ *for all* $\delta > 0$*, it is necessary and sufficient that the analytic continuation of* $\mathbf{f}(\lambda)$ *be an entire function.*

We give the scheme of the proof of Theorems 1 and 2. Under the hypotheses of Theorem 1 there exist constants $0 < a < A < \infty$ such that for all $\lambda \in [-\pi, \pi]$

$$a(\vec{\varphi}, \vec{\varphi}) \leq (\mathbf{f}\vec{\varphi}, \vec{\varphi}) \leq A(\vec{\varphi}, \vec{\varphi}).$$

Therefore, if all elements of the matrix $\mathbf{P}_s(\lambda)$ are polynomials of best approximation of degree $\leq s$ for the corresponding elements of the matrix $\mathbf{f}(\lambda)$, then

$$\left| \int_{-\pi}^{\pi} e^{i\lambda\tau} (\mathbf{f}(\lambda)\vec{\varphi}(\lambda), \vec{\psi}(\lambda))\, d\lambda \right| \leq \int_{-\pi}^{\pi} \left| ((\mathbf{f} - \mathbf{P}_{\tau-1})\vec{\varphi}, \vec{\psi}) \right| d\lambda \leq$$

$$\leq \frac{n}{a} \max_{i,j} E_{\tau-1}(f_{ij}) \left(\int_{-\pi}^{\pi} (\mathbf{f}\vec{\varphi}, \vec{\varphi})\, d\lambda \int_{-\pi}^{\pi} (\mathbf{f}\vec{\psi}, \vec{\psi})\, d\lambda \right)^{1/2}.$$

The last inequality proves Theorem 1 and the sufficiency of the conditions of Theorem 2.

The necessity of the conditions of Theorem 2 is proved according to the following scheme.

1) Choosing in (2) the vectors $\vec{\varphi}(\lambda)$ and $\vec{\psi}(\lambda)$ so that only the k-th component of the vector $\vec{\varphi}(\lambda)$ and the j-th component of the vector $\vec{\psi}(\lambda)$ are nonzero, we obtain

$$\left| \int e^{i\lambda\tau} \varphi_k(\lambda) \psi_j(\lambda) f_{kj}(\lambda)\, d\lambda \right| \leq \alpha(\tau) \left(\int_{-\pi}^{\pi} |\varphi_k(\lambda)|^2 f_{kk}(\lambda)\, d\lambda \cdot \int_{-\pi}^{\pi} |\psi_j(\lambda)|^2 f_{jj}(\lambda)\, d\lambda \right)^{1/2}.$$

From this inequality, by methods similar to those of note (5) (Theorem 3), one derives the smoothness of the functions $f_{kj}(\lambda)$, $k, j = 1, \ldots, n$.

2) Let $\det \mathbf{f}(\lambda_0) = 0$. Let $\mathbf{U}(\lambda_0)$ be the unitary matrix which brings the matrix $\mathbf{f}(\lambda_0)$ to diagonal form. By $\mathbf{Q}_r(\lambda)$ denote the matrix whose off-diagonal elements are zeros, r elements of the main diagonal are $e^{i\lambda} - e^{i\lambda_0}$, and the remaining elements of the main diagonal are 1. By $\mathbf{R}(\lambda)$ denote the diagonal matrix with diagonal elements $(e^{i\lambda} - e^{i\lambda_0})^{-1}$. It turns out that one can choose such a matrix $\mathbf{Q}_r(\lambda)$, $r < n$, that the matrix

$$\mathbf{f}_1(\lambda) = \mathbf{R}\mathbf{Q}_r\mathbf{U}\mathbf{f}\mathbf{U}^*\mathbf{Q}_r^*\mathbf{R}^*(\lambda)$$

will be the s.d. of some stationary process $\tilde{x}(t)$, with $\alpha(\tau; \mathbf{f}_1) = O(\tau^{-r-\beta})$. At the same time the order of the zero λ_0 of the function $\det \mathbf{f}_1(\lambda)$ is at least 2 units less than the order of the same zero of $\det \mathbf{f}(\lambda)$.

3) With the aid of (2) and certain methods of interpolation theory [1], one proves the existence of a trigonometric polynomial $P(\lambda)$ such that

$$\int_{-\pi}^{\pi} |P(\lambda)|^2 \operatorname{sp} \mathbf{f}^{-1}(\lambda)\, d\lambda < \infty.$$

The last inequality means that the total order of the zeros of $\det \mathbf{f}(\lambda)$ is finite. Therefore, by carrying out the construction in item 2) a finite number of times, we ultimately arrive at the representation (4).

3. Let now the process $x(t)$, i.e., the matrix $\mathbf{f}(\lambda)$, have rank $m < n$. Without loss of generality one may assume that rank m is possessed by the matrix

$$\mathbf{g}(\lambda) = \|g_{kj}(\lambda)\| = \|f_{kj}(\lambda)\|, \quad k, j = 1, \dots, m.$$

By $\mathbf{g}^{(pq)}(\lambda)$ we denote the matrix obtained from the matrix $\mathbf{g}(\lambda)$ by replacing its p-th row, $p = 1, \dots, m$, by the row

$$(f_{q1}, \dots, f_{qm}), \quad q = m+1, \dots, n.$$

Finally, put

$$a_{pq}(\lambda) = \det \mathbf{g}^{(pq)}(\lambda) / \det \mathbf{g}(\lambda).$$

The functions $a_{pq}(\lambda)$ determine the relation between the maximal nondegenerate part $(x_1(t), \dots, x_m(t))$ of the process $x(t)$ and the remainder $(x_{m+1}(t), \dots, x_n(t))$. It turns out that complete regularity imposes very stringent restrictions on the relations a_{pq}.

Theorem 5. In order that $\alpha(\tau; \mathbf{f}) \to 0$ as $\tau \to \infty$, it is necessary and sufficient that $\alpha(\tau; \mathbf{g}) \to 0$ as $\tau \to \infty$ and that all functions $a_{pq}(\lambda)$ be rational functions of $e^{i\lambda}$. In this case

$$\alpha(\tau; \mathbf{f}) = O(\alpha(\tau; \mathbf{g}) + e^{-\delta\tau}), \quad \delta > 0.$$

The sufficiency of the conditions of the theorem is verified directly. The proof of necessity is based on the study of the orthogonal complement of the space generated by the vectors

$$(x_1(t), \dots, x_{p-1}(t), x_{p+1}(t), \dots, x_m(t)), \quad -\infty < t < \infty;$$

$$(x_p(t), x_q(t)), \quad |t| > \tau.$$

With the aid of Theorem 5 it is easy to extend Theorems 2–4 to the degenerate case. For example, the analogue of Theorem 2 is the following.

Theorem 6. In order that $\alpha(\tau) = O(\tau^{-\beta})$, it is necessary and sufficient that the matrix $\mathbf{g}(\lambda)$ satisfy the conditions of Theorem 2, and that all functions $a_{pq}(\lambda)$ be rational functions of $e^{i\lambda}$.

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Note: Figure translations are in progress. See original paper for figures.