Soviet-era science, translated into English

CRITICAL WAVELENGTH

PHYSICS

1965

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-196501.67917

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

UDC 523.80

PHYSICS

M. N. MAKSUMOV, L. S. MAROCHNIK

CRITICAL WAVELENGTH

IN COLLISIONLESS GRAVITATING SYSTEMS

(Presented by Academician Ya. B. Zel' dovich on February 8, 1965)

In a gravitating continuous medium, stability is determined by the balance between pressure and gravitation. Therefore there exists a critical size (the Jeans wavelength) that determines the stability boundary of the system. For perturbation sizes exceeding the critical one, pressure does not compensate the gravitational attraction, and instability sets in. It is obvious that a critical wavelength must also exist in systems of gravitating particles (stars) without collisions, owing to the presence of collective interaction. The present note is devoted to the determination of the critical length in collisionless systems.

If the number of particles in the "Debye" sphere is sufficiently large, then a collisionless system of gravitating particles can be described by the Boltzmann and Poisson equations

$$\partial f/\partial t + \mathbf{v}\,\partial f/\partial \mathbf{r} + \nabla\varphi\,\partial f/\partial \mathbf{v} = 0, \tag{1}$$

$$\Delta\varphi = -4\pi G m \int_{(\mathbf{v})} f\,d\mathbf{v},$$

where $f(\mathbf{r},\mathbf{v},t)$ is the distribution function, normalized to the number of particles N in 1 cm^3; φ is the gravitational potential; G is the Newtonian gravitational constant, m is the mass of the particles.

Locally homogeneous problem. Consider small regions of the system in which the unperturbed parameters (f_0, φ_0) do not depend on coordinates or time. Linearizing (1), we obtain the dispersion equation for plane waves $\sim e^{i(\mathbf{kr}-\omega t)}$ (in the form in which the solution of the linearized system is sought) in the form

$$k^2 = -4\pi G m \int_{(\mathbf{v})} \frac{\mathbf{k}\cdot\partial f_0/\partial \mathbf{v}}{\mathbf{kv}-\omega}\,d\mathbf{v}. \tag{2}$$

SovietRxiv

Since we are interested in longitudinal waves, we choose the x-axis in the direction of the wave vector $\mathbf{k}$ and carry out in (2) integration over v_y, v_z. Denoting $\int f_0\, dv_y\, dv_z$ again by f_0, we obtain from (2)

$$k^2 = -4\pi G m \int_{-\infty}^{\infty} \frac{\partial f_0/\partial v}{v-u}\, dv, \tag{3}$$

where $v = v_x$ and $u = \omega/k$. We note that (3), up to sign and a constant factor, coincides with the corresponding dispersion equation in a plasma.

Since u is complex, (3) determines growing ($\operatorname{Im} u > 0$), damped ($\operatorname{Im} u < 0$), and stationary ($\operatorname{Im} u = 0$) waves. Therefore the critical wavelength $\lambda_{\text{cr}} = 2\pi/k_{\text{cr}}$ can be obtained for $\operatorname{Im} u \to 0$ from

upper half-plane of u. In this case

$$k^2 = -4\pi G m \left[\int_{-\infty}^{\infty} \frac{\partial f_0/\partial v}{v-u_1}\, dv + i\pi \left.\frac{\partial f_0}{\partial v}\right|_{v=u_1} \right], \tag{4}$$

where the integral is understood in the sense of the principal value, and $u_1 = \operatorname{Re} u$. Since k is real, it follows that

$$\left.\frac{\partial f_0}{\partial v}\right|_{v=u_1} = 0. \tag{5}$$

The set of roots of equation (5) and $\{u_1^{(\alpha)}\}$ determines the corresponding $\{k_\alpha\}$ by the formula

$$k_\alpha^2 = -4\pi G m \int_{-\infty}^{\infty} \frac{\partial f_0/\partial v}{v-u_1^{(\alpha)}}\, dv. \tag{6}$$

From among the $\{k_\alpha\}$ satisfying (6), it is necessary to single out k_{cr}.

Let us denote the integral in (3) by $W(u)$, i.e.

$$W(u) = \int_{-\infty}^{\infty} \frac{\partial f_0/\partial v}{v-u}\, dv. \tag{7}$$

$W(u)$ carries out a conformal mapping of the upper half-plane u onto some domain D of the W-plane [1]. The roots of (3) that describe growing waves correspond to the intersection of D with the negative real half-axis of the W-plane. Therefore the absence of roots of (3), i.e. the absence of intersection of D with $\operatorname{Re} W < 0$, means stability. Together with the domain D, the axis $\operatorname{Re} W < 0$ intersects its boundary. It is clear that the extreme left point of intersection of the contour of the domain D with $\operatorname{Re} W < 0$ determines, up to a

SovietRxiv

constant factor, $k_{\rm cr}$, since for $k > k_{\rm cr}$ there is no intersection of the domain D with $\operatorname{Re} W < 0$, and the roots of equation (3) are absent. It must be shown that this $k_{\rm cr}$ is among the $\{k_\alpha\}$, i.e. corresponds to that point on the real axis of the u-plane which is an extremum of f_0. $k_{\rm cr}$ is among the $\{k_\alpha\}$, because under a conformal mapping the boundary passes into the boundary; u_1, corresponding to $k_{\rm cr}$, is an extremum of f_0, as is seen from (4). From the $\{k_\alpha\}$ we must choose the largest in magnitude, since $k_{\rm cr}$ is the extreme left point of intersection of the contour of the domain D with $\operatorname{Re} W < 0$.

It is easy to see that the critical k corresponds to that one of the points $\{u_1^{(\alpha)}\}$ at which f_0 has its largest value (the principal maximum). Indeed, (6), with the additional condition (5), can be transformed to the form [2]

$$k_\alpha^2 = -4\pi G m \int_{-\infty}^{\infty} \frac{f_0(v) - f_0(u_1^{(\alpha)})}{\left(v - u_1^{(\alpha)}\right)^2}\, dv. \tag{8}$$

It follows from (8) that k_α is maximal when $u_1^{(\alpha)}$ is the principal maximum of the function f_0. Thus, the formula for determining the critical wavelength has the form

$$\left(\frac{2\pi}{\lambda_{\rm cr}}\right)^2 = k_{\rm cr}^2 = -4\pi G m \int_{-\infty}^{\infty} \frac{\partial f_0/\partial v}{v - u_1}\, dv, \tag{9}$$

where u_1 is the principal maximum of the function f_0.

For example, in a system of particles with a Maxwellian distribution, when

$$f_0 = \frac{N}{\sqrt{\pi}\, u_T}\, e^{-v^2/u_T^2},$$

$k_{\rm cr}$, up to a numerical factor and the form of the characteristic velocity, must coincide with Jeans' s $k_{\rm cr}$. Indeed, $u_1 = 0$ is the principal maximum of such a distribution.

Substituting $u_1 = 0$ in (9), we obtain

$$k_{\rm cr} = \sqrt{8\pi G\rho/u_T^2}, \tag{10}$$

where $\rho = Nm$ is the density of the gas particles. This same result was obtained in (3).

In particular, for all distributions f_0 with a principal maximum at zero, $k_{\rm cr}$ can be found from (3) by putting $\omega = ku = 0$. In this case, integrating (3) by parts, we have

SovietRxiv

$$k_{\text{cr}}^2 = -4\pi G m \left(\overline{\frac{1}{v^2}}\right). \tag{11}$$

This result is due to Ya. B. Zel' dovich.

In the inhomogeneous case, apparently, one cannot indicate an equally simple method for choosing $u_1^{(\alpha)}$, which determines k_{cr}.

The authors express their deep gratitude to Ya. B. Zel' dovich for the idea of this work.

Institute of Astrophysics
Academy of Sciences of the Tajik SSR

Received
8 II 1965

REFERENCES

1. A. I. Akhiezer, G. Ya. Lyubarskii, R. V. Polovin, ZhETF, **40**, 963 (1961).
2. P. Noerdlinger, Phys. Rev., **118**, 879 (1960).
3. D. Lynden-Bell, Monthly Notices RAS, **124**, 279 (1962).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.