ON THE CONVERGENCE OF DIFFERENCE SCHEMES FOR THE WAVE EQUATION WITH VARIABLE COEFFICIENTS

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Abstract

Full Text

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MATHEMATICS

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ON THE CONVERGENCE OF DIFFERENCE SCHEMES FOR THE WAVE EQUATION WITH VARIABLE COEFFICIENTS

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For the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\varkappa(x,t)\frac{\partial u}{\partial x}, \tag{1}$$

where

$$0 < \varkappa_0 \leqslant \varkappa(x,t) \leqslant \chi < \infty, \qquad \varkappa_t'(x,t) \leqslant L < \infty, \qquad 0 \leqslant t \leqslant T < \infty,$$

$\varkappa_0, L, T, \chi$ are constants, we pose the mixed Cauchy problem:

$$u(x,0) = L(x), \qquad u_t'(x,0) = \psi(x), \qquad u(0,t) = u(1,t) = 0. \tag{2}$$

We shall require of the functions $\varkappa(x,t), L, \psi$ such smoothness as would ensure the continuity of the solution of problem (1)–(2), with continuous derivatives up to and including the fourth order.

Let the difference problem corresponding to problem (1)–(2) be

$$(E + \alpha_1 B^{n+1})u^{n+1} + (-2E + \alpha_0 B^n)u^n + (E + \alpha_{-1}B^{n-1})u^{n-1} = 0, \tag{3}$$

where E is the N-dimensional identity matrix;

$$B^n = \left\| -\delta_{i-1}^j r_{i-1}^n + \delta_i^j (r_{i-1}^n + r_i^n) - \delta_{i+1}^j r_i^n \right\|_1^N;$$

δ_i^j is the Kronecker symbol;

$$r_i^n = \frac{\varkappa_{i+1/2}^n \tau^2}{h^2}; \qquad (N+1)h = 1; \qquad i = 1, 2, \dots, N;$$

$$\sum_{s=-1}^{1} \alpha_s = 1; \qquad \alpha_s \geqslant 0; \qquad \varkappa_{i+1/2}^n = \varkappa((i+1/2)h, n\tau);$$

τ is the time step, u^n is an N-dimensional column vector (u_i^n); $\tau/h = \text{const}$;

$$u_i^0 = L(ih), \qquad u_i^1 - u_i^0 = \tau\psi(ih), \qquad 0 \leqslant n\tau \leqslant T. \tag{4}$$

The boundary conditions are taken into account by the matrix B^n and the vector u^n:

$$u_0^n = u_{N+1}^n = 0.$$

Problem (1) with arbitrary $\varkappa(x,t)$ was studied by the method of a priori estimates ([1–3]) and, for $\varkappa = \varkappa_1(x)\varkappa_2(t)$, by the method of separation of variables ([4,5]). In the present note, as in ([6]), the question of convergence is solved by a direct algebraic method, which in the case of nonseparable variables includes noncommutative analysis.

Definition. The difference scheme (3) is called **correct** if, for the root-mean-square norms of the solutions of the equation

$$(E + \alpha_1 B^n)X^2 + (-2E + \alpha_0 B^n)X + (E + \alpha_{-1} B^n) = 0$$

the inequalities

$$\|X\| \leqslant 1 + c\tau$$

hold, where the constant $c > 0$ is independent of τ and n.

Introduce the notation:

$$z = 4(\alpha_0^2 - 4\alpha_1\alpha_{-1})^{-1}, \qquad R = \sup_{x,t} \frac{\tau^2}{h^2}\{\chi(x,t)\}.$$

Denote by D the interval $(0, 4R + \varepsilon)$, where ε is any positive number.

Theorem 1 (convergence criterion). *Let $z \in \overline{D}$. Then the stability of scheme (3) implies convergence of the solution of problem (3)–(4) to the solution of problem (1)–(2) in L_2.*

Let $\chi_1(x,t) \geq \chi_2(x,t) > 0$. Consider two difference schemes of type (3), respectively with $\chi(x,t) = \chi_1(x,t)$ and $\chi(x,t) = \chi_2(x,t)$.

Theorem 2 (comparison theorem). *If scheme (3) is stable for $\chi(x,t) = \chi_1(x,t)$, then it is also stable for $\chi(x,t) = \chi_2(x,t)$. If scheme (3) is unstable for $\chi(x,t) = \chi_2(x,t)$, then it is also unstable for $\chi(x,t) = \chi_1(x,t)$.*

Corollary (local stability criterion). Let

$$M = \sup_{x,t}\{\chi(x,t)\}.$$

Scheme (3) is stable if it is stable for $\chi(x,t) = M$.

The method of investigation carries over directly to boundary-value problems of the second and third kind for the multidimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i=1}^{m} \frac{\partial}{\partial x_i}\chi(x,t)\frac{\partial u}{\partial x_i},$$

and also to equations of the type

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i=1}^{m}\sum_{j=1}^{K_i}(-1)^{j-1}\frac{\partial^j}{\partial x_i^j}\chi_j(x,t)\frac{\partial^j u}{\partial x_i^j}.$$

In all these cases, comparison theorems and local stability criteria have been established.

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Note: Figure translations are in progress. See original paper for figures.