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# NONLINEAR SPINOR FUNCTIONS

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**Abstract**

**Full Text**

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**MATHEMATICS**

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## **NONLINEAR SPINOR FUNCTIONS**

*(Presented by Academician L. I. Sedov, 3 IV 1965)*

**1.** A **spinor of weight  $j$**  and **rank  $r$**  is a quantity that transforms according to the  $r$ -th power of an irreducible representation of weight  $j$  of the full orthogonal group  $W$  of transformations of three-dimensional Euclidean space (<sup>1</sup>). The representation of half-integral weight, which is two-valued, coincides with a representation of the universal covering group of  $W$ , which is single-valued. A quantity transforming according to this representation is a spinor of the corresponding weight.

We shall say that a spinor of weight  $j$  and rank  $r$  **admits a symmetry group**  $G \subseteq W$ , whose representation is realized by some collection of matrices of the representation of weight  $j$  of the group  $W$ , if for each of these matrices the equalities

$$A^{i_1 \dots i_r} = a_{l_1}^{i_1} \dots a_{l_r}^{i_r} A^{l_1 \dots l_r}. \quad (1)$$

hold.

Consider a spinor function  $A(B_1, \dots, B_n)$ , which is a spinor of weight  $j$  and rank  $r$  and depends on  $n$  spinors of weights  $j_1, \dots, j_n$  and ranks  $r_1, \dots, r_n$ . Let the spinor  $B_k$  of weight  $j_k$  and rank  $r_k$  ( $k = 1, \dots, n$ ) admit the symmetry group  $G_k$ ; then the spinor function  $A(B_1, \dots, B_n)$  will admit the symmetry group  $G$ , which is the intersection of the symmetry groups of the arguments  $G_1, \dots, G_n$ . This follows from the fact that the components of the function are functions of the components of the arguments, which admit the symmetry group  $G$ ; therefore the components of  $A(B_1, \dots, B_n)$  also admit the same symmetry group.

This allows us to write the general form of the function, if all invariant linearly independent spinors of weight  $j$  and rank  $r$  with respect to the group  $G$ ,  $J_1, \dots, J_m$ , are known:

$$A = \sum_{p=1}^m k_p J_p, \quad (2)$$

where the scalar coefficients  $k_p$  are functions of the joint scalar invariants of the arguments.

2. Consider the representation of weight  $1/2$ . Knowing the matrices of this representation, one can find, by the method of averaging over the group, spinors of various ranks invariant with respect to subgroups of the group  $W$ , but one can also, by requiring the invariance of certain spinors, obtain from equations (1) the matrix elements  $a_i^i$  of the representation of subgroups of the full orthogonal group  $W$ . It is obvious that any invariant spinor with respect to some group of transformations is a function of spinors specifying this group in the indicated way, and therefore is obtained, by means of the operations of spinor algebra, from its arguments as a spinor function.

The number  $m$  in formula (2) is determined, as is known from the theory of characters <sup>(1)</sup>, by averaging over the group the function  $(a_i^i)^r$ , where  $r$  is the exponent. For odd  $r$ ,  $m = 0$ , since under averaging, along with the terms

the term  $(-a_i^i)^r$  enters, i.e., there are no improper invariants among spinors of rank  $2k - 1$  ( $k = 1, 2, \dots$ ).

Let the spinors  $e_1, e_2$  form a covariant canonical basis; then the fundamental spinor has the form:

$$G = g^{ij}e_i e_j = e_1 e_2 - e_2 e_1.$$

The requirement of its invariance selects, from the set of 4-parameter matrices,

$$\left\| \begin{array}{cc} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{array} \right\|$$

matrices with determinant equal to 1, i.e. we obtain a representation of the groups  $\infty/\infty \cdot m, \infty/\infty$ .

Below, when considering subgroups of the group  $W$ , the invariance of  $G$  will be assumed.

The requirement of invariance of the spinor  $e_1 e_2^2 e_1 + e_2 e_1^2 e_2$  gives a representation of the groups  $m \cdot \infty : m, \infty : 2, \infty \cdot m$ , and the invariance of  $e_1 e_2$ —a representation of the groups  $\infty : m, \infty$ .

Similarly, invariance of  $e_1^2, e_1 e_2$  gives a representation of the groups  $1, \bar{2}; e_1 e_2, e_1^4$ —of the groups  $m, 2, 2 : m; e_1^4 + e_2^4, e_1 e_2^2 e_1 + e_2 e_1^2 e_2$ —of the groups  $2 \cdot m, 2 : 2, m \cdot 2 : m; e_1 e_2, e_1^6$ —of the groups  $3; e_1^6 - e_2^6, e_1 e_2^2 e_1 + e_2 e_1^2 e_2$ —of the groups  $3 : 2; e_1 e_2, e_1^{12}$ —of the groups  $6, 6, 3 : m, 6 : m; e_1 e_2^2 e_1 + e_2 e_1^2 e_2, e_1^6 + e_2^6$ —of the groups  $3 \cdot m; e_1 e_2^6 e_1 + e_2 e_1^6 e_2, e_1^{12} + e_2^{12}$ —of the groups  $6 \cdot m, 6 \cdot m, 6 : 2, m \cdot 3 : m, m \cdot 6 : m; e_1 e_2, e_1^8$ —of the groups  $\bar{4}, 4, 4 : m; e_1 e_2^2 e_1 + e_2 e_1^2 e_2, e_1^8 + e_2^8$ —of the groups  $4 \cdot m, 4 : 2, 4 \cdot m, m \cdot 4 : m$  (all prismatic groups are obtained in the same way).

The condition of invariance of the spinor  $\overline{e_1^5 e_2} - e_1 \overline{e_2^5}$ , where  $\overline{\phantom{x}}$  denotes summation over all isomers, gives

$$\alpha^4 |\alpha|^2 - 5\alpha^4 |\beta|^2 - 5\overline{\beta}^4 |\alpha|^2 + \overline{\beta}^4 |\beta|^2 = 1,$$

$$-6\alpha^5 \overline{\beta} + 6\overline{\beta}^5 \alpha = 0.$$

Solving this system under the condition  $|\alpha|^2 + |\beta|^2 = 1$ , we obtain either

$$\alpha = 0, \quad \beta = e^{i\frac{\pi}{2}k} \quad (k = 1, 2, \dots),$$

or

$$\beta = 0, \quad \alpha = e^{i\frac{\pi}{2}k},$$

or

$$\alpha^4 = \overline{\beta}^4, \quad \alpha = e^{i\frac{\pi}{4}(2k-1)},$$

i.e. a representation of the groups  $\overline{6}/2$ ,  $3/2$ .

Since there is a relation between the “natural” basis of the vector representation (weight 1) and the spinor canonical basis (weight 1/2) <sup>(1)</sup>

$$e_x = \frac{1}{\sqrt{2}}(e_1^2 - e_2^2), \quad e_y = \frac{1}{\sqrt{2}}(e_1^2 + e_2^2), \quad e_z = \frac{1}{\sqrt{2}}(e_1 e_2 + e_2 e_1),$$

the tensors defining the vector representation of the group <sup>(2)</sup>, written in the spinor canonical basis, will be invariant with respect to the spinor representation, which facilitates the finding of invariants; the tensors  $O_h$  and  $Y_h$ , defining respectively the groups  $\overline{6}/4$  and  $3/10$ , have the form:

$$O_h = e_x^4 + e_y^4 + e_z^4,$$

$$Y_h = 2/_{25} [2(5e_x^6 + \overline{e_x^4 e_y^2} + \overline{e_x^2 e_y^4} + 5e_y^6) + (\overline{e_x^5 e_z} + \overline{e_x^3 e_y^2 e_z} + \overline{e_x e_y^4 e_z}) +$$

$$+ (\overline{3e_x^4 e_z^2} + \overline{e_x^2 e_y^2 e_z^2} + \overline{3e_y^4 e_z^2}) + (\overline{e_x^2 e_z^4} + \overline{e_y^2 e_z^4}) + 13e_z^6].$$

3. Let us write the general form of some spinor functions <sup>(2)</sup>.

Groups  $\infty/\infty \cdot m, \infty/\infty$ :  $A^{\alpha\beta} = k_1 g^{\alpha\beta}, \quad A^{\alpha\beta\gamma\delta} = k_1 g^{\alpha\beta} g^{\gamma\delta} + k_2 g^{\alpha\gamma} g^{\beta\delta}.$

Groups  $\infty : m, \infty$ :  $A^{\alpha\beta} e_\alpha e_\beta = k_1 e_1 e_2 + k_2 e_2 e_1, \quad A^{\alpha\beta\gamma\delta} e_\alpha e_\beta e_\gamma e_\delta =$   
 $= k_1 e_1^2 e_2^2 + k_2 e_1 e_2 e_1 e_2 + k_3 e_1 e_2^2 e_1 + k_4 e_2 e_1^2 e_2 + k_5 e_2 e_1 e_2 e_1 + k_6 e_2^2 e_1^2.$

Groups  $m \cdot \infty : m, \infty : 2, \infty \cdot m$ :  $A^{\alpha\beta} = A^{\alpha\beta}(\infty/\infty), \quad A^{\alpha\beta\gamma\delta} e_\alpha e_\beta e_\gamma e_\delta =$

$$= k_1 (e_1^2 e_2^2 + e_2^2 e_1^2) + k_2 (e_1 e_2 e_1 e_2 + e_2 e_1 e_2 e_1) + k_3 (e_1 e_2^2 e_1 + e_2 e_1^2 e_2).$$

Groups  $1, \bar{2}$ : the most general form.

Groups  $m, 2, 2 : m$ :

$$A^{\alpha\beta} = A^{\alpha\beta}(\infty), \quad A^{\alpha\beta\gamma\delta} e_\alpha e_\beta e_\gamma e_\delta = k_1 e_1^4 + k_2 e_2^4 +$$

$$+ A^{\alpha\beta\gamma\delta}(\infty) e_\alpha e_\beta e_\gamma e_\delta.$$

Groups  $2 \cdot m, 2 : 2, m \cdot 2 : m$ :

$$A^{\alpha\beta} = A^{\alpha\beta}(\infty/\infty), \quad A^{\alpha\beta\gamma\delta} e_\alpha e_\beta e_\gamma e_\delta = k_1 (e_1^4 + e_2^4) + A^{\alpha\beta\gamma\delta}(\infty \cdot m) e_\alpha e_\beta e_\gamma e_\delta.$$

Group  $3$ :

$$A^{\alpha\beta} = A^{\alpha\beta}(\infty), \quad A^{\alpha\beta\gamma\delta} = A^{\alpha\beta\gamma\delta}(\infty).$$

Group  $\bar{3} \cdot m$ :

$$A^{\alpha\beta} = A^{\alpha\beta}(\infty/\infty), \quad A^{\alpha\beta\gamma\delta} = A^{\alpha\beta\gamma\delta}(\infty \cdot m).$$

Groups  $\bar{6}, 6, 3 : m, 6 : m$ :

$$A^{\alpha\beta} = A^{\alpha\beta}(\infty), \quad A^{\alpha\beta\gamma\delta} = A^{\alpha\beta\gamma\delta}(\infty).$$

Group  $3 : 2$ :

$$A^{\alpha\beta} = A^{\alpha\beta}(\infty/\infty), \quad A^{\alpha\beta\gamma\delta} = A^{\alpha\beta\gamma\delta}(\infty \cdot m).$$

Groups  $6 \cdot m, \bar{6} \cdot m, 6 : 2, m \cdot 3 : m, m \cdot 6 : m$ :

$$A^{\alpha\beta} = A^{\alpha\beta}(\infty/\infty),$$

$$A^{\alpha\beta\gamma\delta} = A^{\alpha\beta\gamma\delta}(\infty \cdot m).$$

Groups  $\bar{4}, 4, 4 : m$ :

$$A^{\alpha\beta} = A^{\alpha\beta}(\infty), \quad A^{\alpha\beta\gamma\delta} = A^{\alpha\beta\gamma\delta}(\infty).$$

Groups  $4 \cdot m, \bar{4} \cdot m, m \cdot 4 : m, 4 : 2$ :

$$\begin{aligned} A^{\alpha\beta} &= A^{\alpha\beta}(\infty/\infty), & A^{\alpha\beta\gamma\delta} \\ &= A^{\alpha\beta\gamma\delta}(\infty \cdot m). \end{aligned}$$

Groups  $\bar{6}/2, 3/2$ :

$$\begin{aligned} A^{\alpha\beta} &= A^{\alpha\beta}(\infty/\infty), & A^{\alpha\beta\gamma\delta} &= A^{\alpha\beta\gamma\delta}(\infty/\infty), \\ A^{\alpha\beta\gamma\delta\varepsilon\xi} e_\alpha e_\beta e_\gamma e_\delta e_\varepsilon e_\xi &= A^{\alpha\beta\gamma\delta\varepsilon\xi}(\infty/\infty) + k_6(e_1^5 e_2 - e_1 e_2^5). \end{aligned}$$

Groups  $3/\bar{4}, \bar{6}/4, 3/4$ :

$$\begin{aligned} A^{\alpha\beta} &= A^{\alpha\beta}(\infty/\infty), & A^{\alpha\beta\gamma\delta} &= A^{\alpha\beta\gamma\delta}(\infty/\infty), \\ A^{\alpha\beta\gamma\delta\varepsilon\xi} &= A^{\alpha\beta\gamma\delta\varepsilon\xi}(\infty/\infty), & A^{\alpha\beta\gamma\delta\varepsilon\xi\eta\theta} &= A^{\alpha\beta\gamma\delta\varepsilon\xi\eta\theta}(\infty/\infty) + k_{15} O_h^{\alpha\beta\gamma\delta\varepsilon\xi\eta\theta}. \end{aligned}$$

Groups  $3/\bar{10}, 3/5$ :

$$\begin{aligned} A^{\alpha\beta} &= A^{\alpha\beta}(\infty/\infty), & A^{\alpha\beta\gamma\delta} &= A^{\alpha\delta\gamma\beta}(\infty/\infty), \\ A^{\alpha\beta\gamma\delta\varepsilon\xi} &= A^{\alpha\beta\gamma\delta\varepsilon\xi}(\infty/\infty), & A^{\alpha\beta\gamma\delta\varepsilon\xi\eta\theta} &= A^{\alpha\beta\gamma\delta\varepsilon\xi\eta\theta}(\infty/\infty), & A^{\alpha\beta\gamma\delta\varepsilon\xi\eta\theta ix} \\ &= A^{\alpha\beta\gamma\delta\varepsilon\xi\eta\theta ix}(\infty/\infty), & A^{\alpha\beta\gamma\delta\varepsilon\xi\eta\theta ix\lambda\mu} &= A^{\alpha\beta\gamma\delta\varepsilon\xi\eta\theta ix\lambda\mu}(\infty/\infty) + \\ & & & + k_{133} Y_h^{\alpha\beta\gamma\delta\varepsilon\xi\eta\theta ix\lambda\mu}. \end{aligned}$$

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