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Abstract

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THE CARDINALITY AND STRUCTURE OF THE STRUCTURE OF ALL BICOMPACT EXTENSIONS OF A COMPLETELY REGULAR SPACE

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Here two questions are considered concerning the system $K(X)$ of all bicompact (Hausdorff) extensions of a completely regular space X. The first concerns cardinality, and the second—the natural structure of the system $K(X)$ as an ordered (partially) system. The answer to the first question is given by

Theorem 1. *Every infinite cardinal number is the cardinality of the system $K(X)$ for some space X.*

This does not confirm the assertion of Yu. M. Smirnov, expressed by him in posing the question. However, as was recently proved by Banaschewski and Maranda (with the help of proximity spaces) ([1]), *for every pseudocompact space X one always has* $\operatorname{card} K(X) \geq 2 > \aleph_0$.

A partial answer to the second question, prompted by Shirota' s investigations, is given by

Theorem 2. *For every completely regular space X with the first axiom of countability, the system $K(X)$ is a structure if and only if X is locally bicompact.*

As Shirota showed, the system $U_p(X)$ of all precompact uniform structures (proximities) giving the given topology of the space* is (with its natural order) a complete structure if and only if X is locally bicompact. In view of the natural isomorphism between the systems $U_p(X)$ and $K(X)$, in Shirota' s theorem the first can be replaced by the second. Thus, Theorem 2 is a strengthening of the nontrivial half of Shirota' s theorem, though for the case of spaces with the first axiom of countability. The question of how far one may dispense with this axiom here remains open (see below).

§ 1. **Lemma 1.** *For any infinite discrete space D, in the remainder $\beta D \setminus D$ there exists a topological image of the bicompactum βD.*

Proof essentially proceeds by analogy with the countable case described in the book of Gillman and Jerison (see ([2]), p. 90). Partition the set D into

$\aleph_\alpha = \operatorname{card} D$ pairwise disjoint infinite sets A_i. Let $A = \bigcup A_i^{\beta D}$. From the normality of the space D it follows that each set $A_i^{\beta D}$ is both open and closed in βD. Let $x_i \in A_i^{\beta D} \setminus D$ for each i. Since the sets $A_i^{\beta D}$ are pairwise disjoint, the set T of all chosen points x_i is discrete. Clearly, $T \subset \beta D \setminus D$. Every function defined on T can be continuously extended to A, and then, in the case of its boundedness, also to βD. Hence,

$$\overline{T}^{\beta D} = \beta T.$$

It is clear that βT lies in $\beta D \setminus D$ and is homeomorphic to the bicompactum βD, as was required to prove.

Proof of Theorem 1. Let D be an arbitrary infinite discrete space, $X' = \beta D \setminus \overline{T}^{\beta D}$. Then $\beta X' \setminus X' = \beta D$. Let

$$X = \beta X' \setminus D.$$

Since $X' \subset X \subset \beta X'$, we have $\beta X = \beta X'$ and $\beta X \setminus X = D$. Recall–

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* In this note it is assumed throughout that spaces are completely regular, their extensions are bicompact, and mappings are continuous.

that every extension cX of the space X is such a continuous image of the extension βX that the full inverse image $f_c^{-1}x$ of each point x of X consists only of the point x itself, while the full inverse image of every other point is contained in $\beta X \setminus X$. Since D is discrete, all the full inverse images $f_c^{-1}x$ are finite. Let us prove that for every extension cX the number of inverse images consisting of more than one point is also finite. Indeed, if this is not so, then there exist two such sequences of points a_i and b_i that $a_i, b_i \in \beta X \setminus X$, $a_i \neq b_i$, and $f_c a_i = f_c b_i$ for every $i = 1, 2, \ldots$. Let A consist of all the points a_i, and B of all the points b_i. Since $\overline{A}^{\beta X} = \overline{A}^{\beta D}$ and $\overline{B}^{\beta X} = \overline{B}^{\beta D}$, we have $\overline{A}^{\beta X} \cap \overline{B}^{\beta X} = \emptyset$. Let $a \in \overline{A}^{\beta X} \setminus A$. From what precedes it follows that the difference $\beta X \setminus \overline{B}^{\beta X}$ is a neighborhood of the point a. Since $a \in X'$, $f_c^{-1}(f_c a) = a \in \beta X \setminus \overline{B}^{\beta X}$. Hence, by the closedness of the mapping f_c, there is a neighborhood H of the point $f_c a$ such that $f_c^{-1} H \cap \overline{B}^{\beta X} = \emptyset$. But $f_c a \in f_c \overline{A}^{cX}$ and $f_c A = f_c B$. Therefore $b_i \in f_c^{-1} H$ for some i, which is impossible*. From what has been proved it follows that the number of all extensions cX is not greater than $\operatorname{card} D$. Identifying pairs of points of $\beta X \setminus X$ into one, we obtain already $\operatorname{card} D$ distinct extensions of the space X. Consequently, $\operatorname{card} K(X) = \operatorname{card} D$. The theorem is proved**.

Remark 1. The space X occurring in the proof of the theorem is pseudocompact***.

Indeed, every non-pseudocompact space Y contains in $\beta Y \setminus Y$ a nonempty bicompact set of type G_δ. But every such set has cardinality $2^{\mathfrak{c}}$ (see (4), theorems 28 and 49, or (2), p. 132, 9.6).

Remark 2. As was already noted, for every non-pseudocompact space X one always has card $K(X) \geq 2^{\mathfrak{c}}$. This is seen from the inequality $\text{card}(\beta X \setminus X) \geq 2^{\mathfrak{c}}$, which is easily obtained from Remark 1****.

Remark 3. If X is locally bicompact, or if X satisfies the first axiom of countability, then $K(X)$ is either finite or has cardinality $\geq \mathfrak{c}$. Moreover, if βX contains an infinite bicompactum intersecting X in no more than one point, then card $K(X) \geq \mathfrak{c}$.

Remark 4. As Shirota proved (see (5), p. 141, theorem 6), *for every separable metrizable space even the system of all its metrizable extensions has cardinality* $\geq \mathfrak{c}$.

§ 2. As was said above, for every locally bicompact space X the system $K(X)$ is a complete structure*****. Therefore, to prove theorem 2 it is necessary only to show that if X satisfies the first axiom of countability and is not locally bicompact, then $K(X)$ is not even a structure. For this, observe that in this case there is in X a point x such that $x \in X \cap \overline{\beta X \setminus X}$. In view of the first axiom of countability there is in $\beta X \setminus X$ a countable sequence of points x_i converging to x. Hence theorem 2 follows immediately from the following assertion:

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* Our original proof proceeded with the aid of the theory of partitions according to the same scheme.

** For the proof it is important to us only that there exist such a space X that $\beta X \setminus X = D$ and that the closure $\overline{D}^{\beta X}$ is homeomorphic to βD.

*** The spaces X and X' are also interesting from other points of view. For example, X is zero-dimensional, more precisely $\operatorname{ind} X = 0$, and each of its extensions cX is also zero-dimensional ($\operatorname{ind} cX = 0$); moreover the remainder of $cX \setminus X$ is always discrete. Thus, by taking an uncountable D, we obtain a space X that does not have the Lindelöf property (final compactness) in infinity, but is peripherally bicompact, and, in addition, the remainder of every extension cX is zero-dimensionally situated. This gives a partial answer to a question of E. Sklyarenko (see (3), p. 451).

**** For any space we always have $\text{card}(\beta X \setminus X) \leq \text{card}\, K(X) \leq 2^{\text{card}(\beta X)}$.

***** This follows at once from the following facts: 1) the system $K(X)$ has a minimal element (this is the Aleksandrov extension) if and only if X is locally bicompact; 2) for every X, every subsystem of the system $K(X)$ has an upper bound.

Lemma 2. If in the remainder $\beta X \setminus X$ of the space X there exists a countable sequence of points converging to some point of the space X, then the system $K(X)$ is not a lattice.

Proof. Let $\{x_i\}$ be a countable sequence of points of the remainder $\beta X \setminus X$, converging to some point x of the space X. We may assume that the points

x_i are pairwise distinct. Consider the decomposition z_1 of the bicompactum βX, consisting of the pairs $\{x_{2j-1}, x_{2j}\}$, $j = 1, 2, ...$, and of all the remaining points not included in the chosen pairs. If, speaking somewhat imprecisely, we regard as open sets of the decomposition z_1 precisely those open sets of the bicompactum βX which are sums of elements of the decomposition z_1, then we obtain a bicompactum $c_1 X$, which is an extension of the space X. In exactly the same way we obtain also the extension $c_2 X$—one need only consider the decomposition z_2, consisting of the pairs $\{x_{2j}, x_{2j+1}\}$, $j = 1, 2, ...$, and of all the remaining points not included in these pairs. It turns out that there exists no extension cX of the space X which would precede the extensions $c_1 X$ and $c_2 X$.* Indeed, suppose this is not so: let cX be such an extension. Let, further, $y = f_c(x_1)$, where f_c is the natural mapping of the extension βX onto cX. It is not hard to see from the construction and from the commutativity of the diagram

$$\begin{array}{cccc} \beta X & \overset{f_{c_i}}{\longrightarrow} & & c_{iX} \\ & \searrow f_c & \downarrow f_i & (i = 1, 2), \\ & & & cX \end{array}$$

that $f_c(x_i) = y$ for every i, and consequently also $f_c(x) = y$. But $x \in X$. Hence the full inverse image $f_c^{-1} y$ of the point y must consist only of the single point x, which contradicts the equalities obtained earlier. Thus the system $K(X)$ is indeed not a lattice.

Remark 1. There exists a non-locally bicompact space X for which, nevertheless, the system $K(X)$ is a lattice.

Such a space is the space X appearing in the proof of Theorem 1, as is clear from the same properties of the space X which are decisive in the proof mentioned.

Remark 2. The questions concerning the structure of the system $K(X)$ and the structure of the system $U(X)$, consisting of all uniform structures (with their natural partial ordering) inducing on X the given topology, are equivalent. For this purpose we note that (as was said above) there exists an (order) isomorphism of the system $K(X)$ onto the system $U_p(X)$, consisting of all precompact uniform structures taken from $U(X)$. For every structure V from $U(X)$ there exists such a precompact structure V_p from $U(X)$ that $V_p \leq V$. Therefore $K(X)$ is a lattice if and only if $U(X)$ is a lattice (for any X). Finally, let us point out that if by V_β we denote the finest precompact structure from $U(X)$ (it corresponds to the extension βX), then the structure $\inf(V, V_\beta)$ is the greatest of all those precompact structures V_p belonging to $U(X)$ which precede V (and this latter always exists).

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* The extension cX **precedes** the extension c_{iX} if there exists a mapping $f_i : c_{iX} \to cX$ such that $f_i^{-1}(x) = x$ for every x from X.

Note: Figure translations are in progress. See original paper for figures.