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1965

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Abstract

Full Text

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On the Approximate Solution of Certain Classes of Linear Equations

(Presented by Academician N. I. Muskhelishvili on 11 VII 1964)

In the present communication we establish the applicability of one projection method (see [1]) to the linear equation $Ax = y$, when the operator A, acting in a Banach space, is representable in the form of a certain function of a linear isometric operator [2]. This method can be justified in the case when the operator A is invertible at least from one side. As applications, one obtains methods for the approximate solution of one-dimensional singular integral equations on the unit circle, Wiener–Hopf integral equations, their discrete analogues, and some others.

1. Let $\Omega = \Omega(\mathfrak{B})$ be the ring of all linear bounded operators acting in the Banach space $\mathfrak{B}$. An isometric operator $V \in \Omega$ will be called **strictly isometric** if there exists an operator $V^{(-1)} \in \Omega$ such that $V^{(-1)}V = I$, $VV^{(-1)} \neq I$, and $\|V^{(-1)}\| = 1$ (I is the identity operator in $\mathfrak{B}$).

We introduce the following notation: $\mathfrak{R}(V)$ is the linear span of all operators $V^{(j)}$ $(j = 0, \pm 1, ...)$, where $V^{(j)} = V^j$ $(j = -1, -2, ...)$ and $V^{(j)} = (V^{(-1)})^{-j}$ $(j = -1, -2, ...)$; $\mathfrak{R}_+(V)$ and $\mathfrak{R}_-(V)$ are, respectively, the linear spans of the operators V^j $(j = 0, 1, ...)$ and $V^{(j)}$ $(j = 0, -1, ...)$; $\mathfrak{R}(V)$ is the closure (in the norm of the ring Ω) of the linear manifold $\mathfrak{R}(V)$.

To each operator $R = \sum \alpha_j V^{(j)} \in \mathfrak{R}(V)$ we assign the function

$$R(\zeta) = \sum \alpha_j \zeta^j \qquad (|\zeta| = 1).$$

This correspondence is linear and, moreover, for every $R \in \mathfrak{R}(V)$ the relation (see [2])

$$\max |R(\zeta)| \leq \|R\| \qquad (|\zeta| = 1)$$

holds. The latter makes it possible to assign to each operator $A \in \mathfrak{R}(V)$ ($A = \lim R_n$, $R_n \in \mathfrak{R}(V)$; $n = 1, 2, ...$) a function $A(\zeta) = \lim R_n(\zeta)$, continuous on the unit circle. In this case, by definition [2], the operator A is a function $A(\zeta)$ of the operator V: $A = A(V)$.

The ring of functions $A(\zeta)$ $(|\zeta| = 1)$ corresponding to all operators $A \in \mathfrak{R}(V)$ will be denoted by $\mathfrak{R}(\zeta)$. For any Banach space $\mathfrak{B}$, the ring $\mathfrak{R}(\zeta)$ contains all functions representable by absolutely convergent Fourier series. If, however, $\mathfrak{B}$ is a Hilbert space, then $\mathfrak{R}(\zeta)$ coincides with the set of all continuous functions $f(\zeta)$ on the unit circle, and

$$\|f(V)\| = \max_{|\zeta|=1} |f(\zeta)|.$$

In the paper [2] it is established that, in order that an operator $A \in \mathfrak{R}(V)$ be invertible at least on one side, it is necessary and sufficient that

$$A(\zeta) \neq 0 \qquad (|\zeta| = 1). \tag{1}$$

If condition (1) is satisfied, then the operator A will be invertible, invertible only on the left, or invertible only on the right, depending on whether the number

$$\varkappa(A) = \frac{1}{2\pi} [\arg A(e^{i\varphi})]_{\varphi=0}^{\varphi=2\pi} \tag{2}$$

is equal to zero, positive, or negative.

2. Let Λ be some unbounded set of positive numbers, and let P_τ ($\in \Omega,\ \tau \in \Lambda$) be an arbitrary family of projectors converging strongly to the identity operator as $\tau \to \infty$.

If $A \in \Omega$ and the operator $P_\tau A P_\tau$ is invertible in the subspace $P_\tau \mathfrak{B}$, then by $(P_\tau A P_\tau)^{-1}$ we shall denote the operator equal to the inverse of the operator $P_\tau A P_\tau$ on the subspace $P_\tau \mathfrak{B}$ and equal to zero on the subspace $(I - P_\tau)\mathfrak{B}$.

Lemma. Let A ($\in \Omega$) be an invertible operator, and suppose that the operators $P_\tau A P_\tau$, beginning with some τ_0, are invertible in $P_\tau \mathfrak{B}$ and

$$\sup_{\tau > \tau_0} \|(P_\tau A P_\tau)^{-1}\| < \infty.$$

If the operator T ($\in \Omega$) is completely continuous, and the operator $C = A + T$ is invertible, then, beginning with some τ, the operators $P_\tau C P_\tau$ are invertible in $P_\tau \mathfrak{B}$ and the operators $(P_\tau C P_\tau)^{-1}$ converge strongly as $\tau \to \infty$ to the operator C^{-1}.

Suppose that V is a strictly isometric operator with finite defect number $\dim \mathfrak{B}/V\mathfrak{B}$, satisfying the conditions $P_\tau V P_\tau = P_\tau V$, $P_\tau V^{(-1)} P_\tau = V^{(-1)} P_\tau$ ($\tau \in \Lambda$).

Theorem 1. Let condition (1) be fulfilled for the operator $A \in \mathfrak{R}(V)$. Then:

a) if $\varkappa = \varkappa(A) = 0$, then, beginning with some τ, the operators $P_\tau A P_\tau$ are invertible in $P_\tau \mathfrak{B}$ and the operators $(P_\tau A P_\tau)^{-1}$ converge strongly as $\tau \to \infty$ to the operator A^{-1};

b) if $\varkappa > 0$, then, beginning with some τ, the operators $P_\tau V^{(-\varkappa)} A P_\tau$ are invertible in $P_\tau \mathfrak{B}$, and the operators $(P_\tau V^{(-\varkappa)} A P_\tau)^{-1} V^{(-\varkappa)}$ converge strongly as $\tau \to \infty$ to the left inverse of the operator A;

c) if $\varkappa < 0$, then, beginning with some τ, the operators $P_\tau A V^{-\varkappa} P_\tau$ are invertible in $P_\tau \mathfrak{B}$, and the operators $V^{-\varkappa}(P_\tau A V^{-\varkappa} P_\tau)^{-1}$ converge strongly as $\tau \to \infty$ to the right inverse of the operator A.

3. With the aid of Theorem 1 it is easy to obtain an approximate method for solving the Wiener–Hopf equation and its discrete analogue.

1°. Denote by E one of the Banach spaces l_p $(p \geqslant 1)$, c_0, and by V the strictly isometric operator defined in E by the equality

$$V\{\xi_j\}_0^\infty = \{0, \xi_0, \xi_1, \dots\}.$$

The left inverse of V is the operator

$$V^{(-1)}\{\xi_j\}_0^\infty = \{\xi_{j+1}\}_0^\infty.$$

The equation $A\xi = \eta$, where $A \in \mathfrak{R}(V)$, $\xi = \{\xi_j\}_0^\infty$, $\eta = \{\eta_j\}_0^\infty$, in more detailed notation has the form

$$\sum_{k=0}^{\infty} a_{j-k} \xi_k = \eta_j \quad (j = 0, 1, \dots), \tag{3}$$

where the numbers a_j are the Fourier coefficients of the function $A(\zeta) \in \mathfrak{R}(\zeta)$.

Putting

$$P_n\{\xi_j\}_0^\infty = \{\xi_0, \xi_1, \dots, \xi_n, 0, 0, \dots\} \quad (n = 0, 1, \dots),$$

we obtain that, for example, the operator $P_{nAP}n$ is defined in the subspace P_{nE} by the matrix $\|a_{j-k}\|_{j,k=0}^n$. Consequently, the approximate solution of the infinite system reduces to the solution of a finite algebraic system. In the case where $\varkappa(A) = 0$, $A(\zeta)$ expands into an absolutely convergent Fourier series and $E = l_1$, the latter result was obtained by G. Baxter [3].

2°. Let $k(t) \in L_1(-\infty, \infty)$. As shown in [2], the operator A determined by the left-hand side of the Wiener–Hopf equation

$$\varphi(t) - \int_0^\infty k(t-s)\varphi(s)\, ds = f(t) \quad (0 < t < \infty), \tag{4}$$

considered in each of the spaces $L_p(0, \infty)$, $C_0(0, \infty)$, is a function, in the sense indicated above, of the operator*

$$Vf = f(t) - 2\int_0^t e^{s-t} f(s)\, ds, \qquad V^{(-1)}f = f(t) - 2\int_t^\infty e^{t-s} f(s)\, ds.$$

* In the space $L_2(0,\infty)$ the operator V is strictly isometric; in other spaces it is, generally speaking, not such, but Theorem 1 is applicable in these cases as well.

Putting $(P_\tau f)(t) = f(t)$ $(0 < t < \tau)$ and $(P_\tau f)(t) = 0$ $(\tau \leqslant t < \infty)$, we obtain that

$$P_\tau A P_\tau \varphi = \varphi(t) - \int_0^\tau k(t-s)\varphi(s)\,ds \qquad (0 \leqslant t \leqslant \tau).$$

Thus, the approximate solution of equation (4) is reduced to the solution of a Fredholm equation of the second kind. In the space $L_1(0,\infty)$ this result was obtained by I. S. Chebotarev ([4]).

3°. As one more application of Theorem 1, one can obtain the following proposition.

Let $a(\xi)$ $(|\xi| = 1)$ be an arbitrary continuous function satisfying the conditions $a(\xi) \neq 0$ $(|\xi| = 1)$, $\chi(a) = 0$, and let $\{a_j\}_{-\infty}^{\infty}$ be the Fourier coefficients of $a(\xi)$. Then, beginning with some n, the determinants $D_n(a) = \det \|a_{j-k}\|_{j,k=-n}^n$ do not vanish and

$$\lim_{n\to\infty} \frac{D_n(a)}{D_{n-1}(a)} = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log a(e^{i\theta})\,d\theta\right\}.$$

For the case when the function $a(\xi)$ is nonnegative and the functions $a(\xi)$, $\log a(\xi)$ are absolutely integrable, this proposition was proved by Szegő. Under the assumption that $a(\xi)$ and $\log a(\xi)$ belong to the Wiener ring with weight, it was proved by G. Baxter ([3]).

4. We now consider the case when $V \in \Omega$ is an invertible isometric operator. Suppose that in Ω there exists a projector $Q_1(\neq 0, I)$ with respect to which the operator V has the following properties:

$$Q_1 V Q_1 = V Q_1, \qquad Q_1 V^{-1} Q_1 \neq V^{-1} Q_1, \qquad Q_2 V^{-1} Q_2 = V^{-1} Q_2 \quad (Q_2 = I - Q_1).$$

As in the case of a strictly isometric operator, we introduce the rings $\mathfrak{A}_+(V)$, $\mathfrak{A}(V)$, and $\mathfrak{A}(V)$. All the results of item 1, except for the last one, remain valid also for the case considered here.

It is proved in ([2]) that the operator $C = AQ_1 + BQ_2$ $(C = Q_1 A + Q_2 B)$, where $A, B \in \mathfrak{A}(V)$, is invertible at least on one side if and only if

$$A(\xi) \neq 0, \qquad B(\xi) \neq 0 \qquad (|\xi| = 1). \tag{5}$$

If conditions (5) are fulfilled, then for $\chi(A) > \chi(B)$ the operator C is left-invertible, for $\chi(A) < \chi(B)$ the operator C is right-invertible, and for $\chi(A) = \chi(B)$ the operator C is invertible.

Suppose also that: a) the defect number of the operator V in the subspace $Q_1\mathfrak{B}$ is finite; b) the projectors P_τ $(\tau \in \Lambda)$ and Q_1 are permutable; and c) the subspaces $P_\tau Q_1\mathfrak{B}$, $(I - P_\tau Q_1)\mathfrak{B}$, $P_\tau Q_2\mathfrak{B}$, $(I - P_\tau Q_2)$ $(\tau \in \Lambda)$ are respectively invariant with respect to the operators $Q_1 V^{-1}$, VQ_1, $Q_2 V$, $V^{-1}Q_2$. Then the following holds.

Theorem 2. *Let $\mathfrak{B}$ be a Banach space and let A, B be operators from $\mathfrak{A}(V)$ satisfying conditions (5). Then, beginning with some τ, the operators*

$$P_\tau(V^{-\chi_1}AQ_1 + V^{-\chi_2}BQ_2)P_\tau, \quad \text{where } \chi_1 = \chi(A),\ \chi_2 = \chi(B),$$

are invertible, and the operators

$$\left[P_\tau(V^{-\chi_1}AQ_1 + V^{-\chi_2}BQ_2)P_\tau\right]^{-1}$$

tend strongly, as $\tau \to \infty$, to some operator D. If $\chi = \chi_1 - \chi_2 = 0$, then the operator $DV^{-\chi_1}$ is the inverse of the operator $AQ_1 + BQ_2$; if $\chi > 0$ $(\chi < 0)$, then the operator $(V^{-\chi}Q_1 + Q_2)DV^{-\chi_1}$ is a left (right) inverse of the operator $AQ_1 + BQ_2$.

An analogous theorem holds for operators of the form $Q_1 A + Q_2 B$ $(A, B \in \mathfrak{R}(V))$.

Let us note that if $\varkappa_1 \neq 0$, then even in the case $A = B$ the operators $P_\tau(AQ_1 + BQ_2)P_\tau = P_\tau A P_\tau$ may be noninvertible in the subspace $P_\tau\mathfrak{B}$ for all $\tau \in \Lambda$.

5. As a first application of Theorem 2, we indicate a method for the approximate solution of singular integral equations on the circle.

4°. If we put (see (2)) $\mathfrak{B} = L_p(|\xi| = 1)$ $(1 < p < \infty)$, $(Vf)(\xi) = \xi f(\xi)$ $(f \in L_p)$,

$$(Q_1 f)(\xi) = \frac{1}{2}f(\xi) + \frac{1}{2\pi i}\int_{|z|=1} \frac{f(z)}{z - \xi}\,dz, \qquad Q_2 = I - Q_1,$$

then $\mathfrak{R}(V)$ coincides with the ring of functions continuous on the unit circle, and

$$((a(\xi)Q_1 + b(\xi)Q_2)f)(\xi) = \frac{a(\xi) + b(\xi)}{2}f(\xi) + \frac{a(\xi) - b(\xi)}{2\pi i}\int_{|z|=1} \frac{f(z)}{z - \xi}\,dz.$$

The equation $a(\xi)Q_1 f + b(\xi)Q_2 f = g$, written in more detail, has the form

$$\sum_{k=0}^{\infty} a_{j-k} f_k + \sum_{k=-\infty}^{-1} b_{j-k} f_k = g_j \qquad (j = 0, \pm 1, \dots),$$

where a_j, b_j, f_j, g_j are the Fourier coefficients of the functions $a(\xi), b(\xi), f(\xi), g(\xi)$. It is easy to see that the projectors P_n $(n = 1, 2, \dots)$, defined by the equalities

$$P_n f = \sum_{j=-n}^{n} f_j \xi^j,$$

satisfy all the necessary conditions, and the equation $P_n(aQ_1 + bQ_2)P_n f = P_n g$ in the subspace $P_n \mathfrak{B}$ has the form

$$\sum_{k=0}^{n} a_{j-k} f_k + \sum_{k=-n}^{-1} b_{j-k} f_k = g_j \qquad (j = 0, \pm 1, \dots, \pm n). \qquad (6)$$

Under some additional restrictions this result was essentially obtained by V. V. Ivanov ([5]).

6. Theorem 2 makes it possible to obtain approximate methods for solving a paired integral equation (see (2)), the equation transposed to it, and their discrete analogues. In this way one obtains generalizations of results of I. Ts. Gokhberg and V. G. Cheban ([6]) and of I. S. Chebotarev ([4]).

The authors express their gratitude to A. S. Markus for discussion of the results of the present communication.

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Received
14 IV 1964

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.