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Abstract

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ON ONE CLASS OF INVERSE PROBLEMS FOR DIFFERENTIAL EQUATIONS

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By an inverse problem for differential equations is meant the problem of reconstructing the coefficients of an equation from certain characteristics of its solutions. In the present note the investigation of one class of inverse problems for linear differential equations is reduced to the investigation of linear integral equations of the first kind. One special case is studied in detail.

Consider the equation:

$$P_1\left(\frac{\partial}{\partial x_j}\right)u(x,y) = P_3\left(\frac{\partial}{\partial y_j}\right)P_2\left(\frac{\partial}{\partial x_j}\right)u(x,y), \tag{1}$$

where x, y are vectors with components $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$; P_j are polynomials with coefficients continuously dependent on x.

With regard to the polynomial P_1 we assume that the equation

$$P_1\left(\frac{\partial}{\partial x_j}\right)v(x) = 0$$

has a fundamental solution $G(x, x^0)$, defined in the whole space x, x^0*.

Let us formulate the statement of the inverse problem for equation (1) considered in the present note.

Let D_0, D_1 be certain bounded domains of the space x, and suppose that the intersection $D_1 \cap D_0$ is empty. The coefficients of the polynomials P_1, P_2 are given in the whole space x, while the coefficients of the polynomial P_3 are given everywhere outside D_0.

Let, moreover, in the domain D_1 there be given a family of solutions of (1), $u(x, y, \xi)$, depending on the parameter $\xi(\xi_1, \dots, \xi_p)$. The functions $u(x, y, \xi)$ satisfy the conditions:

$$\frac{\partial^r}{\partial y_1^{k_1} \dots \partial y_m^{k_m}} u(x, 0, \xi) = f_{k_1 \dots k_m}(x, \xi),$$

$$\lim_{\rho\to\infty}\int_{\Sigma_{\rho x^0}}\left|\frac{\partial^{r_1}u(x,0,\xi)}{\partial x_1^{i_1}\dots\partial x_n^{i_n}}\right|\cdot\left|\frac{\partial^{r_2}G(x,x^0)}{\partial x_1^{j_1}\dots\partial x_n^{j_n}}\right|d\sigma_x=0, \tag{2}$$

$$\left|\frac{\partial^r u(x,y,\xi)}{\partial y^{k_1}\dots\partial y^{k_m}}\right|\leqslant C_1+|y|^{C_2},$$

where the indices $k_1,\dots,k_m$ run through all values smaller than the maximal values of the corresponding indices in the polynomial $P_3(\partial/\partial y_j)$; $i_1+j_1,\dots,i_n+j_n$ run through an analogous set with respect to the polynomial $P_1(\partial/\partial x_j)$; $\Sigma_{\rho x^0}$ is the sphere of radius ρ with center at the point x^0; $f(x,\xi)$ are given finite functions; C_1, C_2 are certain constants.

It is required to determine the coefficients of the polynomial P_3 in the domain D_0.

* The existence of a fundamental solution has been proved, for example, for elliptic equations of second order and homogeneous regular equations with constant coefficients.

In the case when $n=m=1$, the formulated inverse problem is equivalent to one formulation of the well-known Sturm–Liouville inverse problem for ordinary differential equations.

Denote

$$v(x,\lambda,\xi)=\int_0^\infty u(x,y,\xi)\exp\{-(\lambda,y)\}\,dy$$

(λ is a vector with components $\lambda_1,\dots,\lambda_m$).

By virtue of (1), (2), the function $v(x,\lambda,\xi)$ satisfies the differential equation:

$$P_1\left(\frac{\partial}{\partial x_j}\right)v=P_3(-\lambda_j)\,P_2\left(\frac{\partial}{\partial x_j}\right)v+\psi, \tag{3}$$

where

$$\psi(x,\lambda,\xi)=Q\left(\lambda_j,\frac{\partial}{\partial x_j},\frac{\partial}{\partial y_k}\right)u(x,0,\xi)$$

is the function that arises as a result of integrating by parts the expression for $P_1(\partial/\partial x_j)v$; Q is a certain polynomial.

Multiply both sides of (4) by the function $G(x,x^0)$ and integrate over the whole space x. From inequalities (2), (3) it follows that the integrals will be convergent, and under integration by parts the corresponding substitution terms at infinity will be equal to zero.

Thus we obtain that the function $v(x^0, \lambda, \xi)$ satisfies the integral relation

$$v(x^0, \lambda, \xi) = \int G(x, x^0)\, P_3(\lambda_j)\, P_2\left(\frac{\partial}{\partial x_j}\right) v\, dx + \int G(x, x^0) \psi\, dx. \qquad (4)$$

It follows from (4) that the function $v(x, \lambda, \xi)$ is an analytic function of λ in some neighborhood of the origin.

Denote:

$$v_{k_1 \dots k_m}(x, \xi) = \frac{\partial^r}{\partial \lambda_1^{k_1} \dots \partial \lambda_m^{k_m}}\, v(x, 0, \xi),$$

$$\psi_{k_1 \dots k_m}(x, \xi) = \frac{\partial^r}{\partial \lambda_1^{k_1} \dots \partial \lambda_m^{k_m}}\, \psi(x, 0, \xi);$$

$$P_{k_1 \dots k_m} = \frac{\partial^r}{\partial \lambda_1^{k_1} \dots \partial \lambda_m^{k_m}}\, P_3(0).$$

By virtue of (4), the introduced functions $v_{k_1 \dots k_m}$, $\psi_{k_1 \dots k_m}$, $P_{k_1 \dots k_m}$ satisfy the system of relations

$$v_{k_1 \dots k_m}(x^0, \xi) = \int G(x, x^0) \left\{ \sum_{j_q + i_q = k_q} P^3_{j_1 \dots j_m}\, P_2\left(\frac{\partial}{\partial x_j}\right) v_{i_1 \dots i_m}(x, \xi) \right\} dx + \int G(x, x^0) \psi_{k_1 \dots k_m}(x, \xi)\, dx, \qquad (5)$$

$$v_{0 \dots 0}(x^0, \xi) = \int G(x, x^0) \psi_{0 \dots 0}(x, \xi)\, d\xi.$$

Relations (5) may be regarded as a recurrent system of linear integral equations of the first kind for determining in D_0 the sought functions $P_{j_1 \dots j_m}$, the coefficients of the polynomial P_3. Indeed, by virtue of the last of equalities (5), the function $v_{0 \dots 0}(x^0, \xi)$ may be considered given in the whole space.

Let us now consider the function:

$$v_{0 \dots 1 \dots 0}(x^0, \xi) = \int_{D_0} G(x, x^0)\, P_{0 \dots 1 \dots 0}\, P_2\left(\frac{\partial}{\partial x_j}\right) v_{0 \dots 0}(x, \xi)\, dx +$$

$$+ \int_{D_0} G(x, x^0)\, P_{0 \dots 1 \dots 0}\, P_2\left(\frac{\partial}{\partial x_j}\right) v_{0 \dots 0}(x, \xi)\, dx + \int_{D_2} G(x, x^0) \psi_{0 \dots 1 \dots 0}(x, \xi)\, dx. \qquad (6)$$

If $x^0 \in D_1$, then in equality (6) all terms except the first on the right-hand side may be regarded as given, and, consequently, (6) may be regarded as a linear integral equation of the first kind with respect to the function $P_{\underbrace{0\ldots1\ldots0}_{q}}(x)$, $x \in D_0$, with kernel

$$K(x, x^0, \xi) = G(x, x^0)\, P_2\left(\frac{\partial}{\partial x_j}\right) v_{0\ldots0}(x, \xi), \qquad x^0 \in D_1.$$

If the indicated equation has a unique solution for any q, then, analogously, equations are obtained for determining the functions $P_{\underbrace{0\ldots10}_{q}\underbrace{\ldots1\ldots0}_{q_1}}(x)$, etc.

Let us carry out a detailed investigation of system (5) in the case when the original differential equation has the form

$$\Delta_x^\alpha u(x, y) = \sum_1^\beta a_j(x) \frac{\partial^j}{\partial y^j} u(x, y) \tag{7}$$

(y is a scalar, α, β are integers).

System (5) in the case under consideration takes the form

$$\tilde{v}_k(x^0, \xi) = \int_{D_0} G(x, \xi) v_0(x, \xi) a_k(x)\, dx,$$

$$v_0(x, \xi) = \int G(x, x^0) \psi_0(x, \xi)\, dx, \tag{8}$$

$$\tilde{v}_k(x^0, \xi) = -\sum_1^{k-1} \int G(x, x^0) v_{k-j}(x, \xi) a_j(x)\, dx + v_k(x^0, \xi) -$$

$$- \int_{D_0} G(x, x^0) a_k(x) v_0(x, \xi)\, dx - \int G(x, x^0) \psi_k(x, \xi)\, dx,$$

$$G(x, x^0) = \begin{cases} \gamma_{\alpha n} |x - x^0|^{2\alpha - n} \ln |x - x^0|, & n \text{ even}, \quad 2\alpha - n > 0, \\ \gamma_{\alpha n} |x - x^0|^{2\alpha - n}, & \text{in all other cases.} \end{cases}$$

It follows from (8) that, for uniqueness of the solution of the inverse problem under consideration, it is sufficient that the integral equation of the first kind with respect to the function $a(x)$

$$\int_{D_0} G(x, x^0) v_0(x, \xi) a(x)\, dx = \tilde{v}(x^0, \xi) \tag{9}$$

have a unique solution.

Multiply both sides of (9) by an arbitrary function and integrate over the domain D_1:

$$\int_{D_1} \tilde{v}(x^0, \xi)\rho(x^0)\, dx^0 = \int_{D_0} w(x)v_0(x, \xi)a(x)\, dx,$$

$$w(x) = \int_D G(x, x^0)\rho(x^0)\, dx^0.$$

The function $w(x)$ in (10) is a potential with density $\rho(x^0)$, distributed in the domain D_1; the function $v_0(x, \xi)$ is a potential with density $\psi_0(x, \xi)$, distributed in the domain D_2. Thus, for uniqueness of the solution of (9), it is sufficient that the linear span of the products of the indicated potentials be dense in the set of functions defined in D_0.

Suppose now that the set of functions $\psi_0(x, \xi)$ is such that the set of potentials $v_0(x, \xi)$ is dense in the set of all harmonic functions regular in some extension of the domain $D_0 - D_{0n}$.*

* This condition is satisfied if, for example, $\psi_0(x, \xi) = u(x, 0, \xi) = \delta(x - \xi)$, where the parameter ξ ranges over the whole domain D_2.

The potentials $w(x)$ are dense in the set of all harmonic functions regular in D_{0n}, on the basis of known results of potential theory. It is easy to see that in this case the linear span of products of potentials is dense in the set of continuous functions concentrated in D_0. Indeed, consider harmonic polynomials of degree γ, normalized and homogeneous with respect to the coordinates of the point $x - x^1$, where x^1 is an arbitrary point of D_0.

As is known, the sum of the squares of all the indicated polynomials is equal to $|x - x^1|^2$, while the function $\{1 - |x - x^1|^2/R^2\}^N$ approximates $\delta(x - x^1)$ in the ball $|x - x^1| \le R$.

Thus, we have proved the following uniqueness theorem for the inverse problem for equation (7):

Theorem. *If the family of solutions $u(x, y, \xi)$ of (7) is such that the set of potentials $v_0(x, \xi)$ is dense in the set of all harmonic functions regular in the domain D_{0n} (some extension of the domain D_0), then the solution of the inverse problem for (7) is unique in the class of continuous functions $a_k(x)$ $(k = 1, \dots, \beta)$.*

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Note: Figure translations are in progress. See original paper for figures.