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Abstract

Full Text

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ON A CERTAIN CLASS OF SCATTERING OPERATORS AND CHARACTERISTIC OPERATOR-FUNCTIONS OF CONTRACTIONS

(Presented by Academician V. I. Smirnov on 20 VI 1964)

1°. In questions of scattering theory for two groups of unitary operators U_t and \tilde{U}_t with Lebesgue spectrum, acting in one and the same Hilbert space (t runs through a sequence of integers or the whole real axis), the S -operator of scattering is defined as the product:

$$S(\tilde{U}, U) = W_+(U, \tilde{U})W_-(\tilde{U}, U) \left(= W_+^{-1}(\tilde{U}, U)W_-(\tilde{U}, U) \right), \quad (1)$$

where

$$W_{\pm}(\tilde{U}, U) = s - \lim_{t \rightarrow \pm\infty} \tilde{U}_{-t}U_t$$

are the so-called wave operators.

Below we shall consider unitary groups satisfying the conditions of the scheme of P. Lax and R. Phillips ⁽¹⁾. This scheme makes it possible to define the wave operators and the scattering operator for the case of unitary groups acting in different spaces.

In investigations on the theory of unitary dilations of contraction operators, B. Sz.-Nagy and C. Foias ⁽³⁾ made essential use of the notion of the characteristic operator-function $W(\zeta)$ of a contraction T ,

$$W(\zeta) = -T + \zeta(I - TT^*)^{1/2}(I - \zeta T^*)^{-1}(I - T^*T)^{1/2}, \quad (2)$$

introduced for the first time in this form in ⁽²⁾.

Thanks to the investigations ^(1,3), it is possible to establish a connection between the scattering suboperator and the characteristic operator-function in one class of problems of scattering theory. In the class under consideration, the question of the possibility of reconstructing the group from the S -operator is also clarified, and the analytic properties of the scattering suboperator are studied.

2°. Let $V_t^{\pm}(t \geq 0)$ be two semigroups of isometric operators acting in orthogonal subspaces \mathfrak{D}_{\pm} of the Hilbert space $\mathfrak{H} = \mathfrak{D}_+ \oplus \mathfrak{D}_-$, $V_t^{\pm}\mathfrak{D}_{\pm} \subset \mathfrak{D}_{\pm}$. We shall assume that

$$\bigcap_{t>0} V_t^\pm \mathfrak{D}_\pm = \{0\}. \quad (3)$$

The general case is easily reduced to this one.

A group of unitary operators U_t , acting in a Hilbert space \mathfrak{H}_U , will be called a (unitary) **coupling** of the semigroups V_t^\pm if: 1) $\mathfrak{D}_\pm \subset \mathfrak{H}_U$; 2) $U_{\pm t} f = V_t^\pm f$, $f \in \mathfrak{D}_\pm$ ($t > 0$); 3) $\bigvee U_t \mathfrak{D}_\pm = \mathfrak{H}_U$.

In the discrete case, when t runs through the integers, for the existence of such couplings it is necessary and sufficient that the dimensions

$$m_\pm = \dim(\mathfrak{D}_\pm \ominus V_1^\pm \mathfrak{D}_\pm)$$

be equal.

In the continuous case this condition must be satisfied for the Cayley transforms of the infinitesimal operators of the semigroups V_t^\pm . Indeed,

$$B^\pm = s - \lim_{t \downarrow 0} \frac{1}{it} (V_t^\pm - I)$$

are maximal symmetric operators in the spaces \mathfrak{D}_\pm with defect indices $(0, m_\pm)$. Therefore, for the Cayley transforms

$$V_\pm(\zeta) = (B^\pm - \zeta I)(B^\pm - \bar{\zeta} I)^{-1}$$

of the operators B^\pm , we have

$$V_\pm(\zeta) \mathfrak{D}_\pm \subset \mathfrak{D}_\pm \quad (\text{Im } \zeta > 0), \quad m_\pm = \dim(\mathfrak{D}_\pm \ominus V_\pm(\zeta) \mathfrak{D}_\pm).$$

From (3) it follows for V_t^\pm that

$$\bigcap_k V_\pm^k(\zeta) \mathfrak{D}_\pm = \{0\}.$$

The infinitesimal operator A of an arbitrary coupling U_t of the semigroups V_t^\pm is a self-adjoint extension, with exit into the space \mathfrak{H}_U , of the symmetric operator $B = B^+ \oplus (-B^-)$, whence it follows that the Cayley transform $V(\zeta)$ of the operator A gives a coupling of the semigroups generated by the operators $V_\pm(\zeta)$. Conversely, an arbitrary coupling of the semigroups $V_\pm^k(\zeta)$ gives the Cayley transform of the infinitesimal operator of some coupling of the semigroups V_t^\pm .

Taking into account (3) and properties 1)-3) of the coupling, we obtain that an arbitrary coupling has only Lebesgue spectrum of multiplicity $m = m_\pm$.

For two couplings U_t and \tilde{U}_t of the semigroups V_t^\pm , with exit into the spaces \mathfrak{H}_U and $\mathfrak{H}_{\tilde{U}}$, we define the wave operators by the formulas

$$W_\pm(U, \tilde{U}) = s - \lim_{T \rightarrow \mp\infty} \left(s - \lim_{t \rightarrow \pm\infty} U_{-t} \tilde{U}_t P_{\tilde{U}_T \mathfrak{D}_\pm} \right), \quad (4)$$

where $P_{\tilde{U}_T \mathfrak{D}_\pm}$ are the projectors from $\mathfrak{H}_{\tilde{U}}$ onto $\tilde{U}_T \mathfrak{D}_\pm$.

It is easy to see that the operators introduced exist and $W_\pm f = s - \lim_{t \rightarrow \pm\infty} U_{-t} \tilde{U}_t f$ for $f \in U_T \mathfrak{D}_\pm$. The operators $W_\pm(U, \tilde{U})$ isometrically map $\mathfrak{H}_{\tilde{U}}$ onto \mathfrak{H}_U ,

$$U_t W_\pm(U, \tilde{U}) = W_\pm(U, \tilde{U}) \tilde{U}_t, \quad W_\pm(\tilde{U}, U) = W_\pm^{-1}(U, \tilde{U})$$

and the multiplication theorem holds: for three couplings $U_t^{(1)}, U_t^{(2)}$, and $U_t^{(3)}$ of the semigroups V_t^\pm , the equality

$$W_\pm(U^{(1)}, U^{(3)}) = W_\pm(U^{(1)}, U^{(2)}) W_\pm(U^{(2)}, U^{(3)})$$

is valid.

The scattering operator $S(\tilde{U}, U)$ of the group \tilde{U}_t with respect to the group U_t is defined, as usual, by the product (1) of the wave operators. The operator $S(\tilde{U}, U)$ is unitary on \mathfrak{H}_U and commutes with the group U_t .

If for two couplings U_t and \tilde{U}_t of the semigroups V_t^\pm there exists a coupling $U_t^{(1)}$ such that $S(U, U^{(1)}) = S(\tilde{U}, U^{(1)})$, then for an arbitrary coupling $U_t^{(2)}$ the equality $S(U, U^{(2)}) = S(\tilde{U}, U^{(2)})$ holds. In this case we shall say that the scattering operators of the groups U_t and \tilde{U}_t coincide.

Theorem 1. For two couplings U_t and \tilde{U}_t , the scattering operators coincide if and only if, for all t ,

$$P_U U_t P_U = P_{\tilde{U}} \tilde{U}_t P_{\tilde{U}}, \quad (5)$$

where P_U and $P_{\tilde{U}}$ are the projectors, respectively, from \mathfrak{H}_U and $\mathfrak{H}_{\tilde{U}}$ onto \mathfrak{H} .

In what follows the exposition will be carried out only for discrete groups. The results formulated extend to the case of continuous groups by passing to the Cayley transforms of the corresponding infinitesimal operators. In this connection the following is essential.

Lemma. For any two couplings U_t and \tilde{U} ($-\infty < t < \infty$), the equality

$$W_\pm(U, \tilde{U}) = W_\pm(V(\xi), \tilde{V}(\xi)),$$

holds, where $V(\xi)$ and $\tilde{V}(\xi)$ are the Cayley transforms of the infinitesimal operators of the groups U_t and \tilde{U}_t .

The assertion of the lemma under other conditions was first obtained in [4] (see also [5]).

Since below we shall be speaking only about discrete groups, instead of saying that the group U_t is a coupling of the semigroups V_t^\pm , we shall say that the operator $U = U_1$ is a coupling of the operators $V_\pm = V_1^\pm$.

3°. Let the unitary operator U be a coupling, with exit into the space \mathfrak{H}_U , of the isometric operators V_\pm . Then, if we denote by $\mathfrak{R}_\pm = \mathfrak{D}_\pm \oplus V_\pm \mathfrak{D}_\pm$, the decompositions

$$\mathfrak{D}_\pm = \sum_0^\infty \oplus V_\pm^k \mathfrak{R}_\pm, \quad \mathfrak{H}_U = \sum_{-\infty}^\infty \oplus U^k \mathfrak{R}_\pm. \quad (6)$$

hold. Let $\mathcal{L}_2(-\pi, \pi; \mathfrak{R}_\pm)$ be the Hilbert spaces of vector-functions $f(\lambda)$ with values in \mathfrak{R}_\pm and norm

$$\|f\|^2 = \int_{-\pi}^\pi \|f(\lambda)\|_{\mathfrak{R}_\pm}^2 d\lambda,$$

and let $\mathcal{L}_2^\pm(-\pi, \pi; \mathfrak{R}_\pm)$ be the subspaces of vector-functions from $\mathcal{L}_2(-\pi, \pi; \mathfrak{R}_\pm)$ expandable in a Fourier series respectively in nonnegative and negative powers of $e^{i\lambda}$. Consider the isometric mappings F_\pm of the space \mathfrak{H}_U onto $\mathcal{L}_2(-\pi, \pi; \mathfrak{R}_\pm)$:

$$F_+ f = \sum_{-\infty}^\infty e^{ik\lambda} P_{\mathfrak{R}_+} U^{-k} f, \quad F_- f = \sum_{-\infty}^\infty e^{i(k-1)\lambda} P_{\mathfrak{R}_-} U^{-k} f, \quad f \in \mathfrak{H}_U,$$

where $P_{\mathfrak{R}_\pm}$ are the projectors from \mathfrak{H}_U onto \mathfrak{R}_\pm .

It is easy to see that

$$F_\pm U f = e^{i\lambda} F_\pm f, \quad F_\pm \mathfrak{D}_\pm = \mathcal{L}_2^\pm(-\pi, \pi; \mathfrak{R}_\pm).$$

In [1] an operator of the form

$$\hat{S} = Q F_- F_+^{-1},$$

was introduced, where Q is an arbitrary isometric mapping of \mathfrak{R}_- onto \mathfrak{R}_+ , and was called the scattering operator. It was also shown there that

$$(\hat{S}f)(\lambda) = S(\lambda)f(\lambda), \quad f \in \mathcal{L}_2(-\pi, \pi; \mathfrak{R}_+), \quad (7)$$

where $S(\lambda)$ is the boundary value of an “inner” operator-function $S(\xi)$ on \mathfrak{R}_+ : 1) $S(\xi)$ is analytic inside the disk $|\xi| < 1$; 2) the boundary value (in the strong sense) $S(\lambda)$ is unitary for almost all λ .

One can show that, for an arbitrary “inner” operator-function on \mathfrak{R}_+ , the operator

$$S = F_+^{-1} \hat{S} F_+ = F_+^{-1} Q F_-, \quad (8)$$

where \hat{S} is given by equality (7), is indeed the scattering operator $S(U_0, U)$ for the coupling U under consideration and for some coupling U_0 without exit from the space $\mathfrak{H} = \mathfrak{D}_+ \oplus \mathfrak{D}_-$. Here the operator U_0 is determined uniquely by the S -operator, and $U_0 f = Q f$, $f \in \mathfrak{R}_-$. Conversely, if U_0 is an arbitrary coupling without exit from \mathfrak{H} , then

$$S(U_0, U) = F_+^{-1} U_0 F_-.$$

4°. Let now \tilde{U} be an arbitrary coupling, generally speaking with exit into the space $\mathfrak{H}_{\tilde{U}}$. Then, since $S(\tilde{U}, U)$ commutes with the operator U and is unitary, for the operator $\hat{S} = F_+ S F_+^{-1}$ we obtain

$$(\hat{S}f)(\lambda) = S(\lambda)f(\lambda), \quad f \in \mathcal{L}_2(-\pi, \pi; \mathfrak{R}_+),$$

where $S(\lambda)$ is a unitary operator-function on \mathfrak{R}_+ , called the scattering suboperator. It follows from item 3° that when $\mathfrak{H}_{\tilde{U}} = \mathfrak{H}$, the suboperator $S(\lambda)$ is the boundary value of an “inner” operator-function on \mathfrak{R}_+ .

In the general case the following holds.

Theorem 2. For arbitrary couplings U and \tilde{U} of the operators V_{\pm} , the suboperator $S(\lambda)$ is representable in the form

$$S(\lambda) = S_1^*(\lambda) S_2(\lambda), \quad (9)$$

where $S_1(\lambda)$ and $S_2(\lambda)$ are boundary values of “inner” operator-functions on \mathfrak{R}_+ .

5°. Denote $\mathfrak{K} = \mathfrak{H}_U \ominus \mathfrak{H}$; let $P_{\mathfrak{K}}$ be the projector from \mathfrak{H}_U onto \mathfrak{K} , and introduce the semigroup of contractions on \mathfrak{K}

$$T^n = P_{\mathfrak{K}} U_{\mathfrak{K}}^n P_{\mathfrak{K}}. \quad (10)$$

We shall consider such a coupling U for which $U\mathfrak{N}_- \cap U\mathfrak{N}_+ = \{0\}$. The general case is easily reduced to this one. Then $\overline{(U-T)\mathfrak{K}} = \mathfrak{N}_+$, $\overline{(U^* - T^*)\mathfrak{K}} = \mathfrak{N}_-$, and there exist isometric mappings Q_+ and Q_- of the spaces $\overline{(I - T^*T)^{1/2}\mathfrak{K}}$ and $\overline{(I - TT^*)^{1/2}\mathfrak{K}}$ onto the spaces \mathfrak{N}_+ and \mathfrak{N}_- , such that

$$(U - T)f = Q_+(I - T^*T)^{1/2}f, \quad (U^* - T^*)f = Q_-(I - TT^*)^{1/2}f, \quad f \in \mathfrak{K}.$$

Observe that in the case under consideration U is the minimal unitary dilation of the operator T , and $T^n f \rightarrow 0$, $T^{*n} f \rightarrow 0$ ($n \rightarrow \infty$), $f \in \mathfrak{K}$ (3).

Let $W(\zeta)$ ($|\zeta| < 1$) be the characteristic operator-function (2) of the contraction T . If U_0 is an arbitrary coupling of the operators V_{\pm} without exit from the space \mathfrak{H} , then, according to §§ 3°, 4°, the scattering suboperator of U_0 relative to U is the boundary value of a certain “inner” operator-function $S(\zeta)$ on \mathfrak{N}_+ .

Theorem 3. *The relation holds*

$$S(\zeta) = QQ_-W(\zeta)Q_+^{-1}, \quad (11)$$

where Q is the isometric mapping of \mathfrak{N}_- onto \mathfrak{N}_+ , induced by the operator U_0 .

Conversely, the boundary value of an arbitrary inner function of the form (11) is the scattering suboperator of some coupling without exit relative to the operator U .

6°. If $W(\zeta)$ is the characteristic operator-function of an arbitrary contraction T and ζ_0 is a regular point of the operator T , $|\zeta_0| < 1$, then the operator $W(\zeta_0)$ is continuously invertible and

$$W^{-1}(\zeta_0) = -T^* + (I - T^*T)^{1/2}(\zeta_0 I - T)^{-1}(I - TT^*)^{1/2}.$$

For the class of contraction operators T considered by us, when $W(\zeta)$ is an inner function, Theorem 3 and the results of work (1) imply the converse assertion (compare with (6)).

Moreover, the following theorem is valid, which is more conveniently formulated for the operator-function $S(\zeta)$ connected with $W(\zeta)$ by relation (11).

Theorem 4. *For any ζ , $|\zeta| < 1$, the dimensions of the annihilating subspaces of the operators $T - \zeta I$ and $S(\zeta)$ are equal and*

$$\dim[\mathfrak{K} \ominus \overline{(T - \zeta I)\mathfrak{K}}] = \dim[\mathfrak{N}_+ \ominus \overline{S(\zeta)\mathfrak{N}_+}].$$

A point ζ , $|\zeta| < 1$, is a point of regular type of the operator T if and only if 0 is a point of regular type of the operator $S(\zeta)$.

Couplings of semigroups of isometric operators with exit into different spaces arise in the consideration of the wave equation with different boundary conditions (1).

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