METRIZABILITY, NORMABILITY, AND MULTINORMABILITY OF CONSTRUCTIVE LOCALLY CONVEX SPACES

1965

SovietRxiv

Abstract

Full Text

PHAN ĐÌNH DIỆU

METRIZABILITY, NORMABILITY, AND MULTINORMABILITY OF CONSTRUCTIVE LOCALLY CONVEX SPACES

(Presented by Academician P. S. Novikov, 21 I 1965)

1. In the present note we use the terms and notation introduced in (1,3,4). Judgments are understood in the sense of constructive interpretation (3). Let: a) A_m and A_n be alphabets; b) $\mathfrak{P}$ be a one-parameter formula in a variable α of sort t_m; c) $\mathfrak{E}$ be a two-parameter formula in variables β and γ of sort t_m; d) $\mathfrak{I}$ be a normal one-parameter formula in a variable δ of sort t_n; e) $\mathfrak{R}$ be a three-parameter formula in variables ξ of sort t_n and η, ξ of sort t_m. Suppose that the sets $\mathfrak{P}$ and $\mathfrak{I}$ are nonempty. We agree to denote: by θ and ι subordinate sort letters, whose characteristic formulas are respectively $\mathfrak{P}$ and $\mathfrak{I}$; by $\theta_1, \theta_2, \ldots$ variables of sort θ; by $\iota_1, \iota_2, \ldots$ variables of sort ι. Introduce the notation:

$$(T = U) \Longleftrightarrow F^{\beta,\gamma}_{T,U}[\mathfrak{E}]; \qquad \mathfrak{R}(V, T, U) \Longleftrightarrow F^{\xi,\eta,\xi}_{V,T,U}[\mathfrak{R}],$$

where T and U are arbitrary terms of sort θ; V is an arbitrary term of sort ι.

The list

$$A_m,\ \mathfrak{P},\ \mathfrak{E},\ A_n,\ \mathfrak{I},\ \mathfrak{R} \tag{1}$$

will be called a **constructive uniform space** if the following conditions are satisfied:

I 1. $\forall \theta_1 (\theta_1 = \theta_1)$.
I 2. $\forall \theta_1 \theta_2 \theta_3 (\theta_1 = \theta_2 \mathbin{\&} \theta_1 = \theta_3 \supset \theta_2 = \theta_3)$.
IV 1. $\forall \iota_1 \theta_1 \theta_2 (\theta_1 = \theta_2 \supset \mathfrak{R}(\iota_1, \theta_1, \theta_2))$.
IV 2. $\forall \iota_1 \iota_2 \exists \iota_3 \forall \theta_1 \theta_2 (\mathfrak{R}(\iota_3, \theta_1, \theta_2) \supset \mathfrak{R}(\iota_1, \theta_1, \theta_2) \mathbin{\&} \mathfrak{R}(\iota_2, \theta_1, \theta_2))$.
IV 3. $\forall \iota_1 \exists \iota_2 \forall \theta_1 \theta_2 \theta_3 (\mathfrak{R}(\iota_2, \theta_1, \theta_2) \mathbin{\&} \mathfrak{R}(\iota_2, \theta_2, \theta_3) \supset \mathfrak{R}(\iota_1, \theta_1, \theta_3))$.
IV 4. $\forall \iota_1 \exists \iota_2 \forall \theta_1 \theta_2 (\mathfrak{R}(\iota_2, \theta_2, \theta_1) \supset \mathfrak{R}(\iota_1, \theta_1, \theta_2))$.
IV 5. $\forall \iota_1 \theta_1 \theta_2 \theta_3 \theta_4 (\theta_1 = \theta_3 \mathbin{\&} \theta_2 = \theta_4 \mathbin{\&} \mathfrak{R}(\iota_1, \theta_1, \theta_2) \supset \mathfrak{R}(\iota_1, \theta_3, \theta_4))$.

Words of type θ are called **points** of the space (1); for each word ι_1 of type ι the formula $\mathfrak{R}(\iota_1, \theta_1, \theta_2)$ determines a set of pairs of points of the space (1), which we shall call an **entourage** with index ι_1. The list $(A_n, \mathfrak{I}, \mathfrak{R})$, satisfying IV1-IV5, is called a **uniform structure** of the space. The list $(A_m, \mathfrak{P}, \mathfrak{E})$, satisfying I1-I2, is called a **set with an equality relation**. Thus a uniform space is a set with an equality relation on which a uniform structure is defined.

Let $(A_n, \mathfrak{I}, \mathfrak{R})$ and $(\widetilde{A}_n, \tilde{\mathfrak{I}}, \widetilde{\mathfrak{R}})$ be uniform structures defined on the set $(A_m, \mathfrak{P}, \mathfrak{E})$. We shall say that the structure $(A_n, \mathfrak{I}, \mathfrak{R})$ **majorizes** the structure $(\widetilde{A}_n, \tilde{\mathfrak{I}}, \widetilde{\mathfrak{R}})$ if

$$\forall \tilde{\iota}_1 \exists \iota_1 \forall \theta_1 \theta_2 (\mathfrak{R}(\iota_1, \theta_1, \theta_2) \supset \widetilde{\mathfrak{R}}(\tilde{\iota}_1, \theta_1, \theta_2)),$$

where $\tilde{\iota}_1$ is a variable for words of the set $\tilde{\mathfrak{I}}$. Two structures are called **equivalent** if each of them majorizes the other.

A uniform space (1) (or its uniform structure) is called **T-separable** if

$$\forall \theta_1 \theta_2 (\neg(\theta_1 = \theta_2) \supset \exists \iota_1 \neg \mathfrak{R}(\iota_1, \theta, \theta_2)); \qquad \text{(T)}$$

it is called T'-separable if

$$\forall \theta_1 \theta_2 (\forall l_1 \mathfrak{R}(l_1, \theta_1, \theta_2) \supset \theta_1 = \theta_2). \qquad \text{(T')}$$

Theorem 1. *There exists a constructive uniform space which is T'-separable but not T-separable.*

This theorem is a consequence of Theorem 3 from ([8]).

We shall say that the space (1) (or its uniform structure) has an **enumerable fundamental system of neighborhoods** if the set $\mathfrak{I}$ is algorithmically enumerable.

Let ρ be a metric (respectively, semimetric) function in the set $\mathfrak{P}$ (see ([4]), § 9) such that

$$\forall \theta_1 \theta_2 (\theta_1 = \theta_2 \equiv \rho(\theta_1 \Box \theta_2) = 0). \qquad (2)$$

Then the list $({}_0, \mathfrak{I}, \mathfrak{R})$, where $\mathfrak{I}$ is the set of natural numbers and the formula $\mathfrak{R}$ is defined so that for any i, θ_1, θ_2

$$\mathfrak{R}(i, \theta_1, \theta_2) \equiv (\rho(\theta_1 \Box \theta_2) < 2^{-i}),$$

forms a uniform structure on the set $(A_m, \mathfrak{P}, \mathfrak{E})$, which we shall call the uniform structure corresponding to the metric (respectively, semimetric) function ρ.

The uniform space (1) (or its uniform structure) is called **metrizable** (respectively, **semimetrizable**) if in the set $\mathfrak{P}$ there is a potentially realizable metric (respectively, semimetric) function ρ satisfying (2) and such that the uniform structure corresponding to ρ is equivalent to the structure $(A_m, \mathfrak{I}, \mathfrak{R})$ of the space (1).

Theorem 2. *If the uniform space (1) is metrizable (semimetrizable), then it is separable both in the sense of (T) and in the sense of (T′), and its uniform structure is equivalent to a uniform structure that has an enumerable fundamental system of neighborhoods.*

Theorem 3. *There exists a uniform space which is separable both in the sense of (T) and in the sense of (T′), has an enumerable system of neighborhoods, but is not metrizable (not even semimetrizable).*

The proof of Theorem 3 is based on the theorem that there exists an algorithm Ω of type (nn → n) such that the condition $\mathfrak{T}_l(\Omega(k\Box l) = 0)$ is not algorithmically checkable (see, for example, (2) or (4)).

2. In (8) the concept of a constructive locally convex space was introduced. In this paragraph and in the following paragraphs, the terms and notation introduced in (8) are also used.

Let a constructive linear space be given

$$A_m,\ \mathfrak{P},\ \mathfrak{E},\ +,\ \cdot,\ \mathfrak{D}. \tag{3}$$

We shall say of an algorithm N in $A_m^{ca} \cup {}_3$ that it is a **norm** (respectively, a **seminorm**) in the space (3) if it is an algorithm of type $(\theta \to\ )$ (respectively, of type $(\theta \to\ )$) and satisfies the following conditions:

$$\begin{aligned}
&\text{V 1.}\quad \forall\theta_1(\theta_1 = \mathfrak{D} \equiv N(\theta_1) = 0).\\
&\text{V 2.}\quad \forall a\theta_1(N(a\cdot\theta_1) = M(a)\cdot N(\theta_1)).\\
&\text{V 3.}\quad \forall\theta_1\theta_2(N(\theta_1+\theta_2) \le N(\theta_1) + N(\theta_2)).
\end{aligned}$$

The list

$$A_m,\ \mathfrak{P},\ \mathfrak{E},\ +,\ \cdot,\ \mathfrak{D},\ N$$

will be called a **constructive normed** (respectively, **seminormed**) **space** if $(A_m, \mathfrak{P}, \mathfrak{E}, +, \cdot, \mathfrak{D})$ is a linear space and N is a norm (respectively, a seminorm) in it.

This definition is equivalent to the definition of N. A. Shanin (4).

Let N be a norm (respectively, a seminorm) in the linear space (3). The list $({}_0, \mathfrak{I}, \mathfrak{D})$, where $\mathfrak{I}$ is the set of natural numbers and the for-the formula $\mathfrak{D}$, defined so that $\mathfrak{D}(i_1, \theta_1) \equiv N(\theta_1) < 2^{-i}$, forms a locally convex topology on the space (3), which we shall call the **topology corresponding to the norm** (respectively, **to the seminorm**) N. We shall say that the locally convex space

$$A_m,\ \mathfrak{P},\ \mathfrak{E},\ +,\ \cdot,\ \mathfrak{D},\ A_n,\ \mathfrak{T},\ \mathfrak{D} \tag{4}$$

(or its locally convex topology) is **normable** (respectively, **seminormable**) if in the linear space

$$A_m,\ \mathfrak{P},\ \mathfrak{E},\ +,\ \cdot,\ \mathfrak{D} \tag{5}$$

there is realizable a norm (respectively, seminorm) N such that the topology corresponding to N is equivalent to the topology $(A_n, \mathfrak{T}, \mathfrak{D})$ of the space (4).

Let A_l be an alphabet; let $\mathcal{A}$ be a normal one-parameter formula in a variable of sort t_l, defining a nonempty set of words in A_l; let N be an algorithm of type $(\varkappa\theta \to\ )$ (respectively, of type $(\varkappa\theta \to\ )$), where $\varkappa$ is a subordinate sort letter whose characteristic formula is $\mathcal{A}$. The list $(A_l, \mathcal{A}, N)$ is called a **multinorm** (respectively, a **multiseminorm**) in the linear space (3), if the following conditions are satisfied:

VI 1. $\forall \varkappa_1 \theta_1 (\theta_1 = \mathfrak{D} \supset N(\varkappa_1 \square \theta_1) = 0)$.

VI 2. $\forall \varkappa_1 \theta_1 a\, (N(\varkappa_1 \square a \cdot \theta_1) = M(a) \cdot N(\varkappa_1 \square \theta_1))$.

VI 3. $\forall \varkappa_1 \theta_1 \theta_2\, (N(\varkappa_1 \square \theta_1 + \theta_2) \leq N(\varkappa_1 \square \theta_1) + N(\varkappa_1 \square \theta_2))$.

VI 4. $\forall \varkappa_1 \varkappa_2 \exists \varkappa_3 \forall \theta_1\, (N(\varkappa_3, \theta_1) \geq \max(N(\varkappa_1 \square \theta_1) \square N(\varkappa_2 \square \theta_1)))$.

The list

$$A_m,\ \mathfrak{P},\ \mathfrak{E},\ +,\ \cdot,\ \mathfrak{D},\ A_l,\ \mathcal{A},\ N$$

will be called a **constructive multinormed** (respectively, **multiseminormed**) space if $(A_m, \mathfrak{P}, \mathfrak{E}, +, \cdot, \mathfrak{D})$ is a linear space and $(A_l, \mathcal{A}, N)$ is a multinorm (respectively, multiseminorm) in it.

Let $(A_l, \mathcal{A}, N)$ be a multinorm (respectively, multiseminorm) in the linear space (3). Denote $A^n \rightleftarrows A_l \cup {}_0 \cup \{\tau\}$. Denote by $\mathfrak{T}$ the set of words of the form $n\tau\varkappa_1$, where n is a natural number and $\varkappa_1$ is a word of type $\varkappa$. $\mathfrak{T}$ can be defined by a normal formula. Next construct a formula $\mathfrak{D}$ such that

$$\mathfrak{D}(n\tau\varkappa_1, \theta_1) \equiv (N(\varkappa_1 \square \theta_1) < 2^{-n}).$$

The list $(A_\infty, \mathfrak{T}, \mathfrak{D})$ forms a locally convex topology on the space (3), which we shall call the **topology corresponding to the multinorm** (respectively, multiseminorm) $(A_l, \mathcal{A}, N)$.

We shall say that the locally convex space (4) (or its locally convex topology) is **multinormable** (respectively, **multiseminormable**) if in the linear space (5) there is realizable a multinorm (respectively, multiseminorm) $(A_l, \mathcal{A}, N)$ such that the topology corresponding to $(A_l, \mathcal{A}, N)$ is equivalent to the topology $(A_n, \mathfrak{T}, \mathfrak{D})$ of the space (4).

The locally convex space (4) is called **recursively normable** if it is multinormable by a multinorm $(A_l, \mathcal{A}, N)$, where $\mathcal{A}$ is an algorithmically enumerable set.

Theorem 4. *There exists a locally convex space which is seminormable (multiseminormable), but is not normable (multinormable).*

This theorem is proved with the aid of the theorem that there is no algorithm transforming every F-number into a duplex equal to it ([5,6]).

3. Let a locally convex space (4) be given. One can construct a formula $\mathfrak{R}$ such that, for any words u_1 of type u and θ_1, θ_2 of type θ,

$$\mathfrak{R}(u_1, \theta_1, \theta_2) \equiv \mathfrak{D}(u_1, \theta_1 - \theta_2).$$

Then the list $(A_m, \mathfrak{T}, \mathfrak{R})$ is a uniform structure on the linear space (5). We shall call this uniform structure

uniform structure of the locally convex space (4).

A locally convex space is called **metrizable** (respectively **semimetrizable**) if its uniform structure is metrizable (respectively semimetrizable).

We shall say that a locally convex space (4) (or its topology) has a **countable fundamental system of neighborhoods of zero** if the set $\mathfrak{S}$ is algorithmically enumerable.

Theorem 5. *If a locally convex space $\mathfrak{M}$ is metrizable (semimetrizable), then it is separable both in the sense (T) and in the sense (T′), and its topology is equivalent to a topology that has a countable fundamental system of neighborhoods of zero.*

Theorem 6. *If a locally convex space $\mathfrak{M}$ is countably normed, then it is metrizable.*

Theorem 7. *There exists a locally convex space that is separable both in the sense (T) and in the sense (T′), has a countable fundamental system of neighborhoods of zero, but is not metrizable (not even semimetrizable).*

Theorem 7 is a consequence of Theorem 3. The following question remains open: is every metrizable space countably normed?

4. Let $\mathfrak{A}$ be a subset of the set $\mathfrak{P}$. The set $\mathfrak{A}$ is called **bounded** in the space (4) if

$$\forall l_1 \exists a (a > 0 \,\&\, \forall \theta_1 (\theta_1 \in \mathfrak{A} \supset \mathfrak{D}(l_1, a \cdot \theta_1))).$$

It is easy to see that if a locally convex space is normed (seminormed), then it has a bounded convex neighborhood of zero.

Theorem 8. *There exists a locally convex space that is separable both in the sense (T) and in the sense (T′), has a bounded convex neighborhood of zero, but is not normed (not even seminormed).*

This theorem means that Kolmogorov’ s theorem on normability of linear topological spaces[7] in classical mathematics does not carry over to constructive mathematics.

5. **Theorem 9.** *If a locally convex space $\mathfrak{M}$ is T'-separable, multinormed (respectively multisemnormed), and has a bounded neighborhood of zero, then it is normed (respectively seminormed).*

From Theorems 8 and 9 it follows that

Theorem 10. *There exists a locally convex space that is not multinormed (not even multisemnormed).*

An algorithm $\mathcal{M}$ of type $(l\theta \to\ )$ is called a **Minkowski functional** for the topology $(A_n, \mathfrak{S}, \mathfrak{D})$ if the following conditions are satisfied:

$$\forall l_1, \theta_1 \exists a (a > 0 \,\&\, \mathfrak{D}(l_1, (1 : a) \cdot \theta_1) \supset a \geq \mathcal{M}(l_1 \square \theta_1)),$$

$$\forall l_1 \theta_1 n \exists a (a > 0 \,\&\, \mathfrak{D}(l_1, (1 : a) \cdot \theta_1) \,\&\, a - \mathcal{M}(l_1 \square \theta_1) < 2^{-n}).$$

Theorem 11. *If a Minkowski functional for the topology $(A_n, \mathfrak{S}, \mathfrak{D})$ is potentially realizable, then the space (4) is multinormed.*

Theorem 12. *For every multinormed space $\mathfrak{M}$ one can construct a locally convex topology equivalent to the topology of $\mathfrak{M}$ and such that for it a Minkowski functional is impossible.*

In conclusion, the author expresses deep gratitude to A. A. Markov for supervising the work and for a number of valuable suggestions.

Moscow State University
named after M. V. Lomonosov

Received
5 I 1965

CITED LITERATURE

[1] A. A. Markov, *Tr. Matem. inst. im. V. A. Steklova AN SSSR*, **42** (1954).
[2] A. A. Markov, ibid., **52**, 315 (1958).
[3] N. A. Shanin, ibid., **52**, 226 (1958).
[4] N. A. Shanin, ibid., **67**, 15 (1962).
[5] G. S. Tseitin, ibid., **67**, 295 (1962).
[6] G. E. Mints, *DAN*, **147**, No. 5, 1032 (1962).
[7] A. Kolmogoroff, *Studia Math.*, **5**, 29 (1934).
[8] Fandin Ziev, *DAN*, **162**, No. 4, 46 (1965).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.