ON THE BORCHERS CLASS FOR INTERACTING FIELDS

PHYSICS

1965

SovietRxiv

Abstract

Full Text

UDC 530.145.1

PHYSICS

D. A. SLAVNOV, A. D. SUKHANOV

ON THE BORCHERS CLASS FOR INTERACTING FIELDS

(Presented by Academician N. N. Bogolyubov, March 19, 1965)

1. Consider the theory of one neutral scalar field. Suppose that $A_{in}(x)$ and $A_{out}(x)$ are two complete sets of asymptotic-field operators, related to each other by means of the unitary operator S

$$A_{out}(x) = S^{+} A_{in}(x) S. \tag{1}$$

Assume that there exists an operator $j(x)$ such that

$$A_{out}(x) = A_{in}(x) - \int du\, D(x-u) j(u). \tag{2}$$

Then by the operator of the interpolating field corresponding to the S-matrix defined by (1), we shall mean an operator $A(x)$ of the form

$$A(x) = A_{in}(x) - \int du\, D^{ret}(x-u) j(u). \tag{3}$$

It is said that two interpolating fields $A(x)$ and $B(x)$ correspond to a causal S-matrix and belong to one equivalence class (1) (the Borchers class) if they are mutually local,

$$[A(x), B(y)] = 0 \quad \text{for } x \sim y, \tag{4}$$

at least one of them is local,

$$[A(x), A(y)] = 0 \quad \text{for } x \sim y \tag{5}$$

and complete and, moreover,

$$B_{in}(x) = A_{in}(x) = \varphi(x).$$

Sometimes $^{(1)}$ an S-matrix is called causal if there corresponds to it at least one field satisfying (5). However, such a requirement, in all probability, is not sufficient for defining the concept of causality (see, for example, $^{(2,3)}$). In this connection we shall assume the validity of the stronger condition of causality proposed by N. N. Bogolyubov $^{(4)}$:

$$\frac{\delta}{\delta\varphi(y)}\left(S^{+}\frac{\delta S}{\delta\varphi(x)}\right)=0 \quad \text{for } x \lesssim y. \tag{6}$$

In the theory of free fields Epstein $^{(5)}$ established that the most general expression for a field equivalent to the field $A(x) = A_{in}(x)$ has the form

$$\chi(x)=\sum_{m=1}^{n} P_m(K_x) : A_{in}^{m}(x) :, \tag{7}$$

where n is an arbitrarily large but finite number, $P_m(K_x)$ are finite polynomials in K_x with arbitrary real coefficients, and $P_1(0) = 1$, so that

$$\chi_{in}(x) = A_{in}(x). \tag{8}$$

As for the theory of interacting fields, as Streater and Wightman noted (^{6}), the most general expression for a field belonging to the same equivalence class as the field $A(x)$ probably has a form analogous to (7). Here, however, the question arises: how is one to define the local function of the fields $A(x)$ in the theory of interacting fields?

2. To define the local function of the fields $A(x)$, we shall use the concept, introduced by B. V. Medvedev (^{7}), of the "quasinormal" product of Heisenberg fields

$$N_Q(A(x_1)\dots A(x_n)) \equiv S^{+}T_W(N_n(x_1,\dots,x_n)S), \tag{9}$$

where

$$N_n(x_1,\dots,x_n) \equiv : \varphi(x_1)\dots\varphi(x_n) : . \tag{10}$$

The Wick T-product used in (9) is defined as follows (^{7,8}). The S-matrix must be represented in the form

$$S=\sum_{n=0}^{\infty}\frac{1}{n!}\int dy_1\dots dy_n \Phi_n(y_1,\dots,y_n) : \varphi(y_1)\dots\varphi(y_n) :, \tag{11}$$

after which the chronological ordering of the fields $\varphi(x_i)$ and $\varphi(y_j)$ under the integral sign is to be carried out.

Let us now rewrite the expression $T_W(N_n S)$ in a form more convenient for us:

$$\begin{aligned}T_W(N_n(x_1,\dots,x_n)S) = {} & T_W(N_{n-1}(x_1,\dots,x_{n-1})S)\varphi(x_n)+\\ & + i\sum_{k=1}^{n-1} D^-(x_k-x_n)\,T_W(N_{n-2}(x_1,\dots,x_{k-1},x_{k+1},\dots,x_{n-1})S)-\\ & - i\int du\,D^{ret}(x_n-u)\,T_W\left(N_{n-1}(x_1,\dots,x_{n-1})\frac{\delta S}{\delta\varphi(u)}\right).\end{aligned} \tag{12}$$

Let us note here that the recurrence formula (12) may from the outset be taken as the definition of the operator T_W.

3. Consider the expression

$$A(x) = S^+T_W(\varphi(x)S). \tag{13}$$

In the case of a causal (in the sense of (6)) S-matrix, the field $A(x)$ of the form (13) is an interpolating local field. Indeed, it follows from (12) that

$$A(x) = \varphi(x) - i\int du\,D^{ret}(x-u)S^+\frac{\delta S}{\delta\varphi(u)}, \tag{14}$$

i.e., the field $A(x)$ of the form (13) is representable in the form (3) with

$$j_0(x) = iS^+\frac{\delta S}{\delta\varphi(x)}. \tag{14a}$$

The locality of such a field, proceeding from (6), was proved by us in (^{9}).

Suppose that the field $A(x)$ of the form (13) is complete, and let us attempt, by generalizing formula (13), to obtain the most general expression for a field $B(x)$ belonging to the Borchers class of the field $A(x)$. In the case $S=1$ such a generalization is obvious (^{5}): one need only replace $\varphi(x)$ by $\chi(x)$ of the form (7). We shall prove that, also in the case of interacting fields, the operator B of the form

$$B(x) = S^+T_W(\chi(x)S) \tag{15}$$

is a field belonging to the Borchers class of the field $A(x)$.

First of all we shall prove that the fields $A(x)$ and $B(x)$ are mutually local. For this it is sufficient to show that the operator

$$C_n(x) = S^+T_W(N_n(x)S), \tag{16}$$

obtained from (9) when all the arguments coincide, is local with respect to $A(x)$ (in the sense of (4)). We shall carry out the proof by induction. The case $n = 1$, when $C_n(x) = A(x)$, was considered in (9).

We now consider the case of arbitrary n. It is necessary to prove that

$$R(x, y) \equiv [C_n(x), A(y)] = 0 \quad \text{for } x \sim y. \tag{17}$$

Using (12), for $C_n(x)$ we have the expression

$$\begin{aligned} C_n(x) &= S^+\{T_W(:\varphi^{n-1}(x):S)\varphi(t) + i(n-1)D^-(x-t)T_W(:\varphi^{n-2}(x):S)\}_{t=x} \\ &\quad - i\int du\, D^{ret}(x-u)S^+T_W\left(:\varphi^{n-1}(x):\frac{\delta S}{\delta\varphi(u)}\right). \end{aligned} \tag{18}$$

To prove (17), consider two auxiliary relations (assuming everywhere $x \sim y$). Namely,

$$\begin{aligned} &\left[S^+T_W\left(:\varphi^{n-1}(x):\frac{\delta S}{\delta\varphi(u)}\right), A(y)\right] = \\ &= -i[j_0(u), A(y)]C_{n-1}(x) - [C_{n-1}(x), \delta A(y)/\delta\varphi(u)]; \end{aligned} \tag{19}$$

$$\begin{aligned} &[\{C_{n-1}(x)\varphi(t) + i(n-1)D^-(x-t)C_{n-2}(x)\}, A(y)] = \\ &= -iC_{n-1}(x)\int du\, D(t-u)\delta A(y)/\delta\varphi(u). \end{aligned} \tag{20}$$

Taking (19) and (20) into account, for $x \sim y$ we have

$$R(x, y) = i\int du \Big\{ D^{ret}(x-u)\left[C_{n-1}(x), \frac{\delta A(y)}{\delta\varphi(u)}\right] - D(x-u)C_{n-1}(x)\frac{\delta A(y)}{\delta\varphi(u)} - iD^{ret}(x-u)[j_0(u), A(y)]C_{n-1}(x \tag{21}$$

In the last term of (21) we make a substitution taking (14) into account. Then

$$R(x, y) = i\int du\, D^{adv}(x-u)\left[C_{n-1}(x), \frac{\delta A(y)}{\delta\varphi(u)}\right]. \tag{22}$$

Varying (3) and taking (6) into account, it is not difficult to obtain

$$\delta A(x)/\delta\varphi(y) = 0 \quad \text{for } x \lesssim y. \tag{23}$$

Hence, taking into account the properties of the function D^{adv}, it follows that

$$R(x,y)=0 \quad \text{for } x\sim y. \tag{24}$$

Since the field $B(x)$ of the form (15) is a finite sum of $C_n(x)$ of the form (16), multiplied by polynomials $P_n(K_x)$, it is thereby proved that

$$[B(x),A(y)]=0 \quad \text{for } x\sim y. \tag{25}$$

Using (12), for the field $B(x)$ we obtain the expression

$$B(x)=\chi(x)+\sum_{m=1}^{n}P_m(K_x)\sum_{k=1}^{m}\frac{(-i)^k m!}{(m-k)!k!}\int du_1\dots du_k\times \tag{26}$$

$$\times D^{ret}(x-u_1)\dots D^{ret}(x-u_k)\,S^+\frac{\delta^k S}{\delta\varphi(u_1)\dots\delta\varphi(u_k)}:\varphi^{m-k}(x):.$$

It follows from this that $B_{in}(x)=\chi_{in}(x)$, and taking (8) into account,

$$B_{in}(x)=A_{in}(x). \tag{27}$$

Thus, from (25) and (27), by Borchers' theorem (^{1}), it follows that the field $B(x)$ is an interpolating field belonging to the same equivalence class as the field $A(x)$.

4. Let us note that formula (26) can be given another form if one uses the formula

$$(-i)^n S^+\frac{\delta^n S}{\delta\varphi(x_1)\dots\delta\varphi(x_n)}=K_{x_1}\dots K_{x_n}N_Q(A(x_1)\dots A(x_n)). \tag{28}$$

obtained in (7), and take into account that in many interesting theories, which include all renormalizable theories, the N_Q-product coincides with the Φ-product of fields $A(x)$ introduced by Lehmann. Namely:

$$B(x)=\sum_{m=1}^{n}P_m(K_x)\left\{:\varphi^m(x):+\sum_{k=1}^{m}\frac{m!}{k!(m-k)!}\int du_1\dots du_k\times\right. \tag{29}$$

$$\left.\times D^{ret}(x-u_1)\dots D^{ret}(x-u_k)K_{u_1}\dots K_{u_k}\Phi(A(u_1)\dots A(u_k)):\varphi^{m-k}(x):\right\}.$$

This formula is interesting because with its aid the field $B(x)$ can be expressed only through the fields $A(x)$ and $\varphi(x)$.

Here it is appropriate to note that above we constructed the Borchers class for the field $A(x)$, to which there corresponds the current $j_0(x)$, defined by (14a).

However, in fact we used only two of its properties: a) the causality condition (formula (6)) and b) the integrability condition ([4],[7]) of the form

$$\delta j(x)/\delta\varphi(y) - \delta j(y)/\delta\varphi(x) - i[j(y), j(x)] = 0. \tag{30}$$

Thus, for any field $A(x)$ defined by formula (3), in which the current $j(x)$ satisfies these two conditions, one can construct by formula (29) a field $B(x)$ belonging to the Borchers class for the field $A(x)$.

Further, by Borchers' theorem ([1]), it follows from (25) that the field $B(x)$ itself is local, i.e.,

$$[B(x), B(y)] = 0 \quad \text{for } x \sim y. \tag{31}$$

Since the current $j(x)$ must be a local function of $A(x)$, it is very likely that the expression for it should be sought ([8]) in the form (15), or

$$B(x) = \sum_{m=1}^{n} P_m(K_x) N_Q(A^m(x)), \tag{15a}$$

especially since, as follows from the preceding, the field $B(x)$ possesses not only the property of locality itself, but also the property of causality (in the sense of (6)). In this connection it is of interest whether, having defined the current by an expression of the type (15) or (29) and having somehow fixed the product, one can construct an equation for the field $A(x)$ that has a nontrivial solution. Also remaining unsolved is the question whether the very plausible general expression proposed by us for the field $B(x)$ exhausts the entire equivalence class for the field $A(x)$.

In conclusion, we express our gratitude to B. V. Medvedev and I. T. Todorov for a useful discussion.

Moscow State University
named after M. V. Lomonosov

Moscow Institute of Radio Electronics
and Mining Electromechanics

Received
5 III 1965

CITED LITERATURE

[1] H. J. Borchers, Nuovo Cim., **15**, 784 (1960).

[2] R. Haag, *Les problèmes mathématiques de la théorie quantique des champs*, Lille, 1957; Collection of translations, *Matematika*, **6**, 4, 134 (1962); Suppl. Nuovo Cim., **14**, 131 (1959).

[3] I. T. Todorov, Lectures at the International Winter School of Theoretical Physics, Dubna, March, 1964.
[4] N. N. Bogolyubov, B. V. Medvedev, M. K. Polivanov, *Questions of the Theory of Dispersion Relations*, Moscow, 1958.
[5] H. Epstein, Nuovo Cim., **27**, 886 (1963).
[6] R. F. Streater, A. S. Wightman, *PCT, Spin and Statistics and all that*, 1964.
[7] B. V. Medvedev, M. K. Polivanov, Lectures at the International Winter School of Theoretical Physics, Dubna, March 1964; B. V. Medvedev, *JETP*, **48**, issue 5 (1965).
[8] A. D. Sukhanov, *JETP*, **47**, 1303 (1964); *JETP*, **48**, 538 (1965).
[9] D. A. Slavnov, A. D. Sukhanov, *JETP*, **41**, 1940 (1961).
[10] H. Lehmann, K. Symanzik, W. Zimmermann, Nuovo Cim., **1**, 205 (1955).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.