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MATHEMATICAL PHYSICS

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Abstract

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MATHEMATICAL PHYSICS

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AN INTEGRAL METHOD FOR SOLVING THE GENERAL PROBLEM OF HEAT AND MASS TRANSFER

(Presented by Academician A. Yu. Ishlinskii on 22 IV 1964)

1°. Usually, diffusion and heat-transfer processes are described by equations of parabolic type. In the case when it is necessary to take into account the finite velocity of diffusion, the process is described by an equation of hyperbolic type ([1]). In principle, transfer processes described by other differential or non-differential equations are conceivable. In the present paper a certain integral method is proposed which makes it possible to solve linear transfer problems for various domains under various boundary conditions, under the assumption that the solution of the problem for the whole space R is known. Everywhere below we shall call the transfer process heat conduction and speak of the propagation of heat.

Let D be a certain convex domain with boundary S. The problem is posed of determining the temperature field $G(P, Q, t)$ at the point Q at the time t, if at the point P of the domain at $t = 0$ an instantaneous unit quantity of heat was released, and the surface S is thermally insulated. In solving this problem, in accordance with what was said above, the superposition principle is assumed to hold, and the influence function $\Gamma(P, Q, t)$ for the whole space $R(P, Q \in R)$ is considered known; it may be found either analytically or from the data of a single fundamental experiment.

2°. We first consider the following problem: at $t = 0$, at some point P of the three-dimensional space R, a unit heat impulse arises. Surround the point P by a closed convex surface that does not impede the propagation of heat. We find the heat flux $q(P, M, t)$ at the point M of the surface in the direction of the inward normal. Consider the sphere Ω of radius $r = |PM|$ with center at P. The heat flux q_Ω through its surface is equal to the time derivative of the enthalpy of the sphere

$$q_\Omega(P, M, t) = \frac{\partial}{\partial t} \iiint_\Omega k\Gamma(P, Q, t)\, dQ,$$

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where k is the product of the specific heat by the specific weight. For an isotropic medium,

$$q_\Omega(P,M,t) = 4\pi k \frac{\partial}{\partial t} \int_0^{r=|PM|} \Gamma(\rho,t)\rho^2\, d\rho, \quad \text{where } \rho = |PQ|.$$

Denoting by ν the outward normal at the point M of the surface S, it is not difficult to obtain that

$$q(P,M,t) = k\,\frac{\cos(r,\nu)}{r^2} \int_0^{r=|PM|} \rho^2 \frac{\partial \Gamma(\rho,t)}{\partial t}\, d\rho. \tag{1}$$

$q(P,M,t)$ depends continuously on P and M. Moreover, it is obvious that

$$\int_0^t d\tau \iint_S q(P,M,\tau)\, dM = k \iiint_D \Gamma(P,Q,t)\, dQ - 1;$$

$$\left| \int_0^t d\tau \iint_S q(P,M,\tau)\, dM \right| < 1. \tag{2}$$

Suppose now that the boundary S of the domain D is movable, and that the domain D is convex for all $t \in [0,t_0]$. It is assumed that the motion of the boundary S does not disturb the heat-conduction process in R. The change in the enthalpy of the domain D is ensured both by heat conduction and by displacement of the boundary. Therefore, in this case

$$q(P,M,t) = k\frac{\cos[r(t),v(t)]}{|r(t)|^2} \int_0^{r(t)} \frac{\partial \Gamma(\rho,t)}{\partial t} \rho^2\, d\rho +$$

$$+\, k\cos[r_0(t),v(t)]r_0'(t)\Gamma[P,M(t),t]^*, \tag{3}$$

where $r_0(t) = |OM(t)|$; $r(t) = |PM(t)|$; the point O is the origin of coordinates.

3°. As is known, by virtue of the superposition principle one can immediately write the solution of the heat-conduction problem under boundary conditions of the second kind, if the influence function G for the given domain is known. For example, in the parabolic case, when $T|_{t=0} = f(P)$, the heat flux on the surface is equal to $q(M,t)$, and a function of sources $F(P,t)$ acting in D is prescribed, the temperature $T(P,t)$ is determined as follows:

$$T(P,t) = k \iiint_D f(Q)G(Q,P,t)\, dQ + \int_0^t d\tau \iint_S q(M,\tau)G(M,P,t-\tau)\, dM +$$

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$$+\int_0^t d\tau \iiint_D F(Q,\tau)G(Q,P,t-\tau)\,dQ. \tag{4}$$

Therefore the first main task in the study of heat conduction is to determine the function $G(P,Q,t)$.

To find the function G, one can formulate an integral equation, based on consideration of the mechanism of heat (substance) transfer, which permits one to assert that the action of an adiabatically insulated boundary (a thermal barrier) will be manifested in the "reflection" of the flux $q_v(P,M,t)$ (1) into the domain D. In other words, the function G is the result of the superposition of heat sources of intensity $(-q)$, distributed over S, plus the source function for infinite space. The minus sign takes account of the fact of reflection. This makes it possible, in accordance with (4), to formulate the integral equation

$$G = -\int_0^t d\tau \iint_S q(P,M,t-\tau)G(M,Q,\tau)\,dM + \Gamma(P,Q,t). \tag{5}$$

Equation (5) has a unique solution, which can be found by the method of successive approximations, setting

$$\varphi_0 = \Gamma(P,Q,t);$$

$$\varphi_n(P,Q,t) = (-1)^n \int_0^t d\tau \iint_S q(P,M,t-\tau)\varphi_{n-1}(M,Q,\tau)\,dM.$$

Then

$$G = \varphi_0 + \varphi_1 + \varphi_2 + ... + \varphi_n + ... \tag{6}$$

The uniform and absolute convergence of the series (6) can be established by majorizing it by a geometric progression with ratio less than 1. In doing so, estimate (2) and the negativity of q for domains of the indicated class are used.

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* If the moment τ of the appearance of the thermal charge does not coincide with the initial time of reckoning $(0 \le \tau \le t)$, then in the expression for q, instead of $\partial\Gamma(\rho,t)/\partial t$ and $\Gamma[P,M(t),t]$, one should substitute, respectively, $\partial\Gamma(\rho,t-\tau)/\partial t$ and $\Gamma[P,M(t),t-\tau]$.

It is interesting to note that the terms of the series (6) admit the following very intuitive physical interpretation. The initial heat pulse propagates in the region

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D in the same way as in the whole space, leaving on each element dS of the surface in a neighborhood of the point M, during the time $d\tau$, the heat charge $[-q(P, M, \tau)dS\, d\tau]$, which we shall call the first reflection. This charge, in turn, by virtue of the principle of independence of action, also propagates as in the space R, leaving on the boundary elements $d\sigma$ in a neighborhood of the point N at the instant ξ, during the time $d\xi$, the charge

$$[-q(P, M, \tau)dS\, d\tau][-q(M, N, \xi - \tau)d\sigma\, d\xi] \quad (\tau \leqslant \xi \leqslant t),$$

which we shall call the second reflection, and so on. These reflections give the corresponding additions to the temperature field $\Gamma(P, Q, t)$, which have the form:

$$1) \quad -\int_0^t d\tau \iint_S q(P, M, \tau)\Gamma(M, Q, t - \tau)\, dM;$$

$$2) \quad \int_0^t d\tau \iint_S q(P, M, \tau)\, dM \int_\tau^t d\xi \iint_S q(M, N, \xi - \tau)\Gamma(N, Q, t - \xi)\, dN.$$

It is not difficult to show that the functions thus obtained coincide with the corresponding terms of the series (6). Let us note that the solution (6) of equation (5) satisfies the condition of conservation of heat, i.e.

$$k \iiint_D G(P, Q, t)\, dQ = 1.$$

This is easily shown by integrating equation (5) over the region D, using equality (2) and the uniqueness of the solution.

4°. On the basis of what was set forth in item 2°, one can find the function $G(P, Q, t, \tau)$ for a region with a moving boundary. In this case the problem is formulated as follows. At a certain point P of the region D of variable volume (mass), at the instant τ, a heat pulse of unit intensity arises. The increment of mass (we consider the case of expansion of the region D) has zero excess temperature. The boundary is thermally insulated. Find $G(P, Q, t, \tau)$. Characteristic for the present problem is the presence of a moving boundary and, hence, the circumstance that the temperature field G depends not on the difference $t - \tau$, but on t and τ, because the determining factor will be not only the duration of action $t - \tau$, but also the instant τ at which the pulse arises.

We extend $\Gamma(P, Q, \xi - \tau)$ $(\tau \leqslant \xi \leqslant t)$ to the newly arising layer of matter, assigning to it the temperature zero, each time as $\Gamma[P, M(\xi), \xi - \tau] - \Gamma[P, M(\xi), \xi - \tau]$. After this, the first reflection on the element dS during the time $d\xi$ will be the sum of two terms. The first is the quantity of heat, taken with a minus sign,

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which should have entered $D(\xi)$ by heat conduction through the element dS during the time $d\xi$, i.e. it is constructed as the first reflection for a fixed region (see item 3°). The second term is the enthalpy of the layer of matter that has arisen on dS during the time $d\xi$ at the instant ξ as a result of the motion of the boundary $S(\xi)$, at temperature $\{-\Gamma[P, M(\xi), \xi - \tau]\}$. Consequently, the reflection flux is equal to $\{-q[P, M(\xi), \tau, \xi]\}$ (see (3)).

Integrating the equation, we obtain, proceeding from the fact that $G(P, Q, t, \tau)$ in this case should be regarded as the result of superposition of heat sources of intensity $(-q)$ distributed on S and moving together with S, plus the function Γ for the whole space:

$$G(P, Q, t, \tau) = -\int_0^t d\xi \iint_{S(\xi)} q[P, M(\xi), \tau, \xi] G[M(\xi), Q, t, \xi]\, dM(\xi) + \Gamma(P, Q, t-\tau). \tag{7}$$

It is not difficult to establish the convergence of the successive-approximation series for equation (7).

5°. The solution of the first boundary-value problem of transport consists in determining the temperature field $T(P, t)$ in D (a region with a fixed boundary), if the initial conditions, the boundary condition $T(P, t)\big|_{P \in S} = \varphi(P, t)$, and the function of sources $F(P, t)$ acting in D are given. For simplicity, let us take zero initial conditions and $F(P, t) = 0$. Since $G(P, Q, t)$ for D is found from (5), then, on the basis of (4), in order to solve the problem it is necessary to find the heat flux q that realizes this process.

Suppose that at some instant t the boundary condition is satisfied. To realize the required boundary condition at the subsequent instant, it is necessary to impart to the surface S a flux consisting of two parts: $q = q_1 + q_2$. The first part of the flux, q_1, serves to maintain the temperature field $\varphi(P, t)$ at the previous level, i.e., it neutralizes the intrinsic rate of temperature equalization at the boundary. The second part $q_2 = \bar{k}\partial\varphi(Pt)/\partial t$ is needed for the required change of $\varphi(P, t)$ in time.

According to (4),

$$q_1 = -\bar{k} \int_0^t d\tau \iint_S (q_1 + q_2) \frac{\partial G}{\partial t}\, dS.$$

The minus sign takes account of the fact of neutralization. Hence

$$q(M, t) = -\bar{k} \int_0^t d\tau \iint_S q(N, \tau) \frac{\partial G(N, M, t-\tau)}{\partial t}\, dN + \bar{k} \frac{\partial \varphi(M, t)}{\partial t}. \tag{8}$$

Here

$$\bar{k} = 1 / \iint_S G(N, M, 0)\, dN.$$

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The terms of the series of successive approximations for equation (8) can be given a certain physical meaning, just as was done in 3°. It is easy to see that q_1 (the neutralizing flux) is an infinite series of successive compensations. The first compensation is the neutralization of the rate of temperature equalization created by the flux q_2; the second compensation is the neutralization of the influence of the flux of the first compensation, etc. The possibility of constructing such a physical model gives grounds to suppose that the series of successive approximations for equation (8) in the problem under consideration will converge. The convergence of this series in some interval $[0, t]$, depending only on the domain, is proved rigorously.

In an analogous manner one solves the first boundary-value problem for a region with a moving boundary (the Stefan problem).

6°. Let now a boundary condition of the third kind be prescribed on the surface, i.e., the law of heat exchange and the temperature of the surrounding medium. Suppose that the boundary of the body is fixed, the initial condition is homogeneous, and take the heat-exchange law in the form of Newton' s law of cooling $q = \alpha(T_{\text{okr}} - T_{\text{gr}})$; α is the heat-transfer coefficient. We seek the solution in the form

$$\int_0^t d\tau \iint_S qG\, dS.$$

Making use of the boundary condition, we obtain:

$$q = -\alpha \int_0^t d\tau \iint_S qG\, dS + \alpha T_{\text{okr}}. \tag{9}$$

It is easy to prove that the series of successive approximations converges absolutely and uniformly for any t.

In conclusion, we note that, using the results given above, one can solve transport problems with moving sources, contact thermal conductivity, and others, both with zero and with nonzero initial conditions. In addition, the results set forth extend to the case of nonconvex regions.

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CITED LITERATURE

1. F. M. Morse, G. Feshbach, *Methods of Theoretical Physics*, 1, IL, 1958.

Note: Figure translations are in progress. See original paper for figures.

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