SOME ASYMPTOTIC PROPERTIES OF ORTHOGONAL POLYNOMIALS IN THE COMPLEX DOMAIN

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Abstract

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SOME ASYMPTOTIC PROPERTIES OF ORTHOGONAL POLYNOMIALS IN THE COMPLEX DOMAIN

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Polynomials orthogonal on a contour were first defined by G. Szegő in 1921. The principal asymptotic properties of these polynomials under various conditions on the weight function and the contour were investigated by G. Szegő, V. I. Smirnov, P. P. Korovkin, and Ya. L. Geronimus (^{1,2}). In the present note the case of multiple smoothness of the contour and of the weight is considered.

The construction of a system of orthogonal polynomials corresponding to a given weight function is usually carried out by means of the familiar process of orthogonalizing the system of powers of the independent variable; moreover, the orthogonal polynomials obtained as a result of such a process are easily expressed in terms of the power moments of the weight function (^{1,2}). But for the investigation of the asymptotic properties of orthogonal polynomials it is expedient to orthogonalize not the system of powers $\{z^n\}$, but a certain sequence of polynomials $\{B_n(z)\}$, defined depending on the weight function and the contour.

Let the polynomials $\{P_n(z)\}$ be orthonormal with weight $n(z)$ on a closed contour Γ bounding a simply connected domain G. By analogy with the formulas mentioned above, which contain power moments, here we shall have the representation

$$P_n(z) = \frac{1}{\sqrt{\Delta_{n-1}\Delta_n}} \begin{vmatrix} h_{00} & h_{01} & \dots & h_{0n} \\ h_{10} & h_{11} & \dots & h_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ h_{n-1,0} & h_{n-1,1} & \dots & h_{n-1,n} \\ B_0(z) & B_1(z) & \dots & B_n(z) \end{vmatrix},$$

where

$$h_{km} = \frac{1}{2\pi} \int_\Gamma n(z) B_k(z) \overline{B_m(z)}\, |dz|, \tag{2}$$

and the determinant $\Delta_n = |h_{km}|_0^n$ is positive, since the quadratic form corresponding to it, by formula (2), is positive definite.

The moments $\{h_{km}\}$ will be called **normal moments** of the weight function $n(z)$ and the contour Γ, if the conditions

$$h_{km} = \delta_{km} + \varepsilon_{km}, \qquad \delta_{km} = \begin{cases} 1, & k = m, \\ 0, & k \neq m, \end{cases} \tag{3}$$

are satisfied, where the numbers $\{\varepsilon_{km}\}$ are such that the double series with general term $|\varepsilon_{km}|^2$ converges.

As the polynomials $\{B_n(z)\}$ one may choose generalized Faber polynomials, whose weight function $g(z)$ is related to the weight $n(z)$ by the condition

$$n(z) = \left|\sqrt{\Phi'(z)}\, g(z)\right|^2, \qquad z \in \Gamma, \tag{4}$$

where the function $w = \Phi(z)$ maps conformally the exterior D of the contour Γ onto the exterior of the unit disk under the conditions $\Phi(\infty) = \infty$ and $\gamma = \Phi'(\infty) > 0$.

The function $g(z)$, analytic in the domain D, is determined uniquely by condition (4), if one assumes that $a_0 = g(\infty) > 0$.

We shall say that the curve Γ belongs to the class $C(p, \alpha)$, where p is a nonnegative integer and $0 < \alpha < 1$, if in the equation of the curve $z = z(s)$, where s is arc length, the function $z(s)$ is continuously differentiable p times, and $z^{(p)}(s) \in \operatorname{Lip} \alpha$.

Lemma 1. *If the function $g(z)$ is continuously differentiable p times in the closed domain $\overline{D}$, with $g^{(p)}(z) \in \operatorname{Lip}\alpha$, and $\Gamma \in C(p'+1, \alpha')$, where $p' + \alpha' > p + \alpha + \frac{1}{2}$, then for the Faber polynomials the formula*

$$B_n(z) = g(z)\Phi^n(z) + O\left(\frac{\ln n}{n^{p+\alpha}}\right), \qquad z \in \overline{D}. \tag{5}$$

holds.

This formula under the conditions $p' = p + 1$ and $\alpha' = \alpha$ was established in our paper (6). Under the conditions of Lemma 1 the proof is unchanged.

Lemma 2. *If the weight function $g(z)$ of the Faber polynomials $\{B_n(z)\}$ satisfies condition (4), then under the conditions of Lemma 1 formula (3) and the estimate*

$$|\varepsilon_{km}| \leq \frac{c_1}{(k+1)^{p+\alpha}(m+1)^{p+\alpha}}. \tag{6}$$

hold.

The assertions of Lemma 2 follow from the known properties of generalized Faber polynomials, set out in our paper (6).

Lemma 3. *If the conditions of Lemmas 1 and 2 are satisfied, and* $p+\alpha>\frac{1}{2}$*, then for the determinant*

$$\Delta_n(n,k)=\begin{vmatrix} 1+\varepsilon_{00} & \varepsilon_{01}\cdots\varepsilon_{0,k-1} & \varepsilon_{0,k+1}\cdots\varepsilon_{0n} \\ \varepsilon_{10} & 1+\varepsilon_{11}\cdots\varepsilon_{1,k-1} & \varepsilon_{1,k+1}\cdots\varepsilon_{1n} \\ \dots\dots\dots\dots & & \\ \varepsilon_{k,0} & \varepsilon_{k,1}\cdots\varepsilon_{k,k-1} & \varepsilon_{k,k+1}\cdots\varepsilon_{kn} \\ \dots\dots\dots\dots & & \\ \varepsilon_{n-1,0} & \varepsilon_{n-1,1}\cdots\varepsilon_{n-1,k-1} & \varepsilon_{n-1,k+1}\cdots\varepsilon_{n-1,n} \end{vmatrix}, \tag{7}$$

obtained from Δ_n *by deleting the elements of the* $(n+1)$*-st row and the* $(k+1)$*-st column, the estimate*

$$|\Delta_n(n,k)|\le c_2/(n+1)^{p+\alpha}(k+1)^{p+\alpha} \tag{8}$$

holds.

Indeed, multiply all elements of the last column of determinant (7) by $(n+1)^{p+\alpha}$ and apply Hadamard' s inequality, summing the squares of the elements by rows. Since in the $(k+1)$-st row the element $1+\varepsilon_{kk}$ is absent, taking into account inequality (6) and the condition $p+\alpha>\frac{1}{2}$, we obtain

$$\left|(n+1)^{p+\alpha}\Delta_n(n,k)\right|^2\le$$

$$\le\frac{c_3}{(k+1)^{2p+2\alpha}}\prod_{m=0}^{n}\left[1+(n+1)^{2p+2\alpha}|\varepsilon_{m,n}|^2+2|\varepsilon_{m,m}|+\sum_{s=0}^{n}|\varepsilon_{m,s}|^2\right],$$

and inequality (8) is proved.

Theorem 1. *If the function* $n(z)$ *is positive and continuously differentiable* p *times, with* $n^{(p)}(z)\in\operatorname{Lip}\alpha$ *and* $p\ge1$*, and* $\Gamma\in C(p'+1,\alpha')$*, where* $p'+\alpha'>p+\alpha+\frac{1}{2}$*, then the formula*

$$P_n(z)=g(z)\Phi^n(z)\left[1+O\left(\frac{\ln n}{n^{p+\alpha}}\right)\right],\qquad z\in\overline{D}. \tag{9}$$

holds.

Proof. Expanding determinant (1) with respect to the elements of the last row, we obtain

$$P_n(z)=\sqrt{\frac{\Delta_{n-1}}{\Delta_n}}\,B_n(z)+\frac{(-1)^{n+1}}{\sqrt{\Delta_{n-1}\Delta_n}}\sum_{k=0}^{n-1}(-1)^k\Delta_n(n,k)B_k(z). \tag{10}$$

As is known (6), the leading coefficient of the Faber polynomial is $a_0\gamma^n$, and for the leading coefficient μ_n of the orthogonal polynomial $P_n(z)$ the formula (4,5)

$$a_0^2\gamma^{2n}/\mu_n^2 = 1 + O(1/n^{2p+2\alpha}), \tag{11}$$

holds, with the aid of which, using (1) or (10), we find that the quantity Δ_n tends to a positive limit. Consequently, by inequality (8) and the condition $p \geqslant 1$, the second term in formula (10) decreases for $z \in \Gamma$ with rate $n^{-p-\alpha}$. Since this term is a polynomial of degree not higher than $(n-1)$, by means of formulas (5) and (11) we obtain formula (9).

The convergence of the sequence $\{\Delta_n\}$ to a positive limit can be proved by means of the uniqueness property of Faber series, and then formula (11) can be obtained by means of Lemma 3.

It is not difficult to show that the estimate of the remainder term in formula (9) is sharp in the sense of order.

Formula (9) is an improvement of our result (^{4,5}), in which instead of $\ln n$ there was the factor $\sqrt{n}$ under the condition $p \geqslant 0$. From formula (10) and inequality (8) it follows that if $p = 0$ and $\alpha > 1/2$, then the remainder term of the asymptotic formula for the orthogonal polynomial will decrease with rate $n^{1-2\alpha}$.

Theorem 2. If the function $p(z)$ is positive and differentiable on Γ, with $p'(z) \in \operatorname{Lip}\alpha$, and $\Gamma \in C(2, \alpha')$, where $\alpha' > 1/2$, then there exists a constant c_4 such that, for every function $f(z)$ analytic in the domain G and continuous in the closed domain, the inequality

$$\left| f(z) - \sum_{n=0}^{N} a_{nP}n(z) \right| \leqslant c_4 E_N(f, \overline{G}) \ln N, \qquad z \in \overline{G},$$

holds, where $\{a_n\}$ are the Fourier coefficients of the function $f(z)$ with respect to the orthonormal system $\{P_n(z)\}$, and $E_N(f, \overline{G})$ is the best uniform approximation of the function $f(z)$ in the closed domain $\overline{G}$ by polynomials of degree not higher than N.

This theorem is proved with the aid of formula (9) for $p = 1$.

The method set forth for investigating asymptotic properties of orthogonal polynomials can also be applied to one particular case of orthogonality over an area. We give without proof analogues of Theorems 1 and 2.

Theorem 3. If $\Gamma \in C(p+1, \alpha)$, where $p + \alpha > 3/2$, then for the polynomials $\{K_n(z)\}$, orthonormal over the area of the domain G with weight $h(z) \equiv 1$, the asymptotic formula

$$K_n(z) = \sqrt{\frac{n+1}{\pi}}\, \Phi'(z)\Phi^n(z) \left[1 + O\left(\frac{1}{n^{p+\alpha-1/2-\varepsilon}}\right)\right], \qquad z \in \overline{D}, \tag{12}$$

holds, where $p + \alpha \geqslant 3/2 + \varepsilon$ and the constant in the estimate of the remainder term depends on $\varepsilon > 0$.

Theorem 4. If $\Gamma \in C(2, \alpha)$, where $\alpha > 1/2$, then there exists a constant c_5 such that, for every function $f(z)$, analytic in the domain G and continuous in the closed domain, the inequality

$$\left| f(z) - \sum_{n=0}^{N} b_{nK} n(z) \right| \leqslant c_5 E_N(f, \overline{G}) \ln N, \qquad z \in \overline{G},$$

holds, where $\{b_n\}$ are the Fourier coefficients of the function $f(z)$ with respect to the orthonormal system of polynomials $\{K_n(z)\}$.

Formula (12) in the case of an analytic contour Γ and with another estimate of the remainder term was first obtained by T. Carleman (^{1,2}).

The idea of introducing normal moments satisfying condition (3) arises in connection with a general theorem on the asymptotic properties of specially constructed determinants, formulated in the work of P. Rosenbloom and S. Warschawski (^{3}). In that work the case of orthogonality with respect to area is considered under the condition $h(z) \equiv 1$, and formula (12) is given without an estimate of the remainder term, but only with the assertion that it tends to zero.

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