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Abstract

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SHORT-WAVE ASYMPTOTICS OF THE EIGENFUNCTIONS OF THE HELMHOLTZ EQUATION

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In the present note, by the method of the parabolic equation, asymptotic formulas are found for the eigenvalues $k_{p,q}$ and eigenfunctions $U_{p,q}(x,y)$ of the problem

$$\Delta U + k^2 U = 0, \qquad U|_S = 0 \quad (\partial U/\partial n|_S = 0)\,,$$

where S is a closed, smooth, convex contour bounding a plane domain D. The method of the parabolic equation ([1]) makes it possible to obtain asymptotic formulas, valid for $kR \gg 1$, where R is a characteristic size of the domain D, only for those eigenfunctions which oscillate rapidly along some narrow strip of the domain D and decrease exponentially in the perpendicular direction outside this strip. Eigenfunctions of this type were first discovered in ([2]), based on the geometrical concept of rays.

1. The directions of the fastest oscillations of the eigenfunctions may be defined as the directions in a neighborhood of which the system of rays arising as a result of multiple reflections is stable in the first approximation. We shall refine this notion of stability for two different cases: when the aforementioned system of rays is close to the contour S, and when it is close to the minimal diameter of the domain D.

Let us consider the first case. Let a ray coinciding with the chord N_0N_1 of the domain D form, with the tangent to S at the point N_0, the angle ε_0. As a result of successive reflections of the ray N_0N_1, the rays $N_1N_2, N_2N_3, \dots$ arise, which form, with the tangents to S at the points $N_1, N_2, \dots$, the angles $\varepsilon_1, \varepsilon_2, \dots$. We shall call the system of rays $N_0N_1, N_1N_2, N_2N_3, \dots$ **stable in the first approximation** if, for any fixed number of reflections m, one can specify a $\delta(m) > 0$ such that, for $|\varepsilon_0| < \delta(m)$, the relative deviation of the ray $|\varepsilon_m \varepsilon_0^{-1}|$ does not exceed a certain constant depending only on the contour S.

It can be shown that, for a smooth strictly convex twice continuously differentiable contour,

$$\varepsilon_m = a(m)\varepsilon_0 + O(\varepsilon_0^2) \quad (\varepsilon_0 \to 0),$$

where $O(\varepsilon_0^2)$ depends on m. The preceding definition of stability in the first approximation reduces to the requirement $|a(m)| < K$, where K depends only on S. Under the indicated restrictions on S, this requirement is fulfilled, and consequently the system of rays under consideration is stable in the first approximation.

Let us investigate the second case. Let $2d$ be the length of an extremal diameter Γ of the domain D; ρ_1 and ρ_2 the radii of curvature of the contour S at the points of its intersection with Γ. We choose Γ as the OY axis and place the origin at the midpoint of Γ. Let

$$x = \alpha y + \beta d$$

be the equation of a ray deviating little from Γ, and let the matrix A describe, in the linear approximation, the change in the ray parameters α and β after two reflections. The definition of stability is analogous to the first case (the role of the small parameter ε is played by $\sqrt{\alpha^2 + \beta^2}$). The system of rays under consideration will be stable in the first approximation if the eigenvalues λ_1 and λ_2 of the matrix A satisfy the condition $|\lambda_1| \le 1,\ |\lambda_2| \le 1$. (For

$\lambda_1 = \lambda_2$ it is necessary additionally to require that $A = A(*)$. The indicated conditions are satisfied only for a relatively minimal diameter $(\rho_1 + \rho_2 > 2d)$ under one of the following restrictions:

$$\text{either} \quad 2d > \rho_1,\ 2d > \rho_2; \qquad \text{or} \quad 2d < \rho_1,\ 2d < \rho_2; \qquad \text{or} \quad 2d = \rho_1 = \rho_2. \tag{1}$$

2. We shall obtain asymptotic formulas for the eigenvalues and eigenfunctions associated with the system of rays arising in a neighborhood of the contour S. Introduce coordinates n and s, where n is the magnitude of the normal to the contour S; s is the length of the arc, measured from some initial point to the foot of the normal. For points inside the contour $n < 0$. The coordinates n, s uniquely determine the position of a point inside the domain D under the condition $n > -\rho_{\min}$, where $\rho_{\min}$ is the minimum value of the radius of curvature $\rho(s)$ of the contour S. We regard the quantity $M = \rho^{1/3}(s)(k/2)^{1/3}$ as a large parameter $(M \gg 1)$. Eigenfunctions of this type will be sought in the form

$$U(x, y) = \operatorname{Re} e^{iks} W(n, s; k),$$

where the "attenuation function" $W(n, s; k)$ must satisfy the conditions

$$\text{I. } W(0, s; k) = 0 \ \left(\text{or } \partial W/\partial n\big|_{n=0} = 0\right).$$

$$\text{II. } e^{ikL}W(n, s+L; k) = W(n, s; k),$$

where L is the length of the contour S (the condition of periodicity of $U(x, y)$ with respect to the variable s).

Introduce the variables [3]:

$$v = 2\left(\frac{k}{2}\right)^{2/3}\rho^{-1/3}(s)n, \qquad \sigma = \left(\frac{k}{2}\right)^{1/3}\int_0^s \rho^{-2/3}(t)\,dt.$$

For the function

$$\Psi(v, \sigma) = \rho^{-1/6}(s)\exp\left[\frac{i}{12}\left(\frac{\rho k}{2}\right)^{-1/3}\frac{d\rho}{ds}v^2\right]W(n, s; k)$$

we obtain the equation

$$i\frac{\partial\Psi}{\partial\sigma} + \frac{\partial^2\Psi}{\partial v^2} + v\Psi + \frac{1}{M^2}\{...\} = 0. \tag{2}$$

Neglecting in equation (2) terms of order $1/M^2$, we obtain an equation with separable variables, whose solution has the form

$$\Psi(v, \sigma) = Ae^{it\sigma}v(t - v),$$

where

$$v(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty \cos\left(\frac{1}{3}y^3 + xy\right)dy$$

is the Airy function. We choose the solution that decreases as $v \to -\infty$.

Condition I determines a discrete set of values of the separation constant $t = t_p$, $p = 1, 2, ...$ (or $t = t'_p$), where t_p (or t'_p) are the zeros of $v(x)$ (or $v'(x)$) [4]: $t_1 = -2.33$; $t_2 = -4.08\,...$; ... ($t'_1 = -1.01\,...$; $t'_2 = -3.24\,...$; ...). Condition II leads to an equation; solving it with respect to k, we obtain a discrete set of eigenvalues

$$k = k_{p,q} = \frac{2\pi q}{L} - t_p\left(\frac{\pi q}{L}\right)^{1/3}\frac{1}{L}\int_0^L \rho^{-2/3}(t)\,dt + ...\,, \tag{3}$$

where $q \gg 1$ and is an integer. Having determined $k_{p,q}$ for the eigenfunctions $U_{p,q}$, we obtain the asymptotic formula

$$U_{p,q} = \frac{A}{\rho^{1/6}(s)} v \left[t_p - \frac{2}{\rho^{1/3}(s)} \left(\frac{k_{p,q}}{2} \right)^{2/3} n \right] \times$$

$$\times \cos \left\{ k_{p,q} s + t_p \left(\frac{k_{p,q}}{2} \right)^{1/3} \int_0^s \rho^{-2/3}(t)\, dt + \frac{1}{6} \frac{d \ln \rho(s)}{ds} k_{p,q} n^2 \right\}. \quad (4)$$

For $n > n_{p,q} = t_p \dfrac{\rho^{1/3}(s)}{2} \left(\dfrac{k_{p,q}}{2} \right)^{-2/3}$ the eigenfunctions $U_{p,q}$ oscillate; for $n < n_{p,q}$ they decrease exponentially. The applicability conditions for formulas (3), (4) may be written in the form

$$\pi q \frac{\rho_{\min}}{L} \gg 1, \qquad -c\rho^{1/3}(s) \left(\frac{\pi q}{2L} \right)^{-2/3} < n \leq 0, \qquad |t_p| < 2c,$$

where c is a constant independent of M (of the order of several units).

3. Let us turn to the consideration of the second case. Let $2d$ be the length of a relatively minimal diameter of the domain D, and let ρ_1 and ρ_2 satisfy conditions (1). Instead of the previously chosen Cartesian system XOY, introduce the elliptic coordinate system ξ, η:

$$x = a \operatorname{ch} \xi \sin \eta, \qquad y - y_0 = a \operatorname{sh} \xi \cos \eta, \qquad -\infty < \xi < \infty, \qquad |\eta| < \pi/2. \quad (5)$$

We choose the parameters a and y_0 so that at the points of intersection with the minimal diameter the contour S has first-order contact with the coordinate ellipses $\xi = \xi_1$ and $\xi = \xi_2$. The parameters a, y_0, ξ_1, and ξ_2 turn out to be real only when condition (1) is satisfied, and the formulas hold

$$a = \frac{1}{|4d - \rho_1 - \rho_2|} \sqrt{2d(\rho_1 + \rho_2 - 2d)(2d - \rho_1)(2d - \rho_2)},$$

$$y_0 = d - a \operatorname{sh} \xi_1,$$

$$\operatorname{sh} \xi_1 = \sqrt{\frac{2d}{\rho_1 + \rho_2 - 2d} \frac{2d - \rho_2}{2d - \rho_1}}, \qquad \operatorname{sh} \xi_2 = \sqrt{\frac{2d}{\rho_1 + \rho_2 - 2d} \frac{2d - \rho_1}{2d - \rho_2}}.$$

We shall now regard the quantity $M = \sqrt{2ka} \gg 1$ as the large parameter. We seek the eigenfunctions in the form

$$U = A^+ e^{ika \operatorname{sh} \xi} W_+(\xi, \eta, k) + A^- e^{-ika \operatorname{sh} \xi} W_-(\xi, \eta; k),$$

where the "attenuation function" $W_\pm(\xi,\eta,k)$ must satisfy the condition

$$\text{I.}\quad U\big|_{\xi=\xi_1} = U\big|_{\xi=\xi_2} = 0 \quad \left(\text{or}\ \ \partial U/\partial\xi\big|_{\xi=\xi_1} = \partial U/\partial\xi\big|_{\xi=\xi_2} = 0\right).$$

Pass to the variables (5) $\tau = \sqrt{2ka}\sin\eta,\ \zeta = \arcsin\operatorname{th}\xi$ and to the new unknown function $\Psi_\pm(\tau,\zeta) = \cos^{-1/2}\zeta \cdot W_\pm$. For the function $\Psi_\pm(\tau,\zeta)$ we obtain the equation

$$\frac{\partial^2\Psi_\pm}{\partial\tau^2} \pm i\frac{\partial\Psi_\pm}{\partial\zeta} - \frac{\tau^2}{4}\Psi_\pm + \frac{1}{M^2}\{...\} = 0. \tag{6}$$

Neglecting in equation (6) terms of order $1/M^2$, we obtain the equation of the harmonic oscillator and take its solution which decreases as $|\tau| \to \infty$:

$$\Psi_\pm(\tau,\zeta) = \exp[\mp i(q+1/2)\zeta]\,\exp(-\tau^2/4)H_q(\tau/\sqrt{2}),$$

where q is an integer;

$$H_q(x) = (-1)^q e^{x^2}\frac{d^q}{dx^q}e^{-x^2}$$

are the Hermite polynomials.

Condition I leads to a homogeneous linear system for the coefficients $A_\pm$. Equating the determinant of this system to zero gives an equation for the eigenvalues $k_{p,q}$, solving which we find*

$$k_{pq} = \frac{\pi p}{2d} + \frac{(q+1/2)}{2d}\arccos\sqrt{\left(1-\frac{2d}{\rho_1}\right)\left(1-\frac{2d}{\rho_2}\right)}, \tag{7}$$

* Formula (7) coincides with the formula for the "eigenvalues" of an open resonator obtained in [6] when diffra

where $p \gg 1$ and is an integer. For the eigenfunctions $U_{p,q}$ we obtain the asymptotic formula

$$U_{p,q} = A\exp\left[-\frac{k_{p,q}a}{2}\sin^2\eta\right]H_q\left(\sqrt{k_{p,q}a}\sin\eta\right)\operatorname{ch}^{-1/2}\xi\times$$

$$\times\begin{matrix}\cos\\ \sin\end{matrix}\left\{k_{p,q}a\operatorname{sh}\xi - \left(q+\frac{1}{2}\right)\arcsin\operatorname{th}\xi - \frac{1}{2}\varphi_{p,q}\right\}, \tag{8}$$

where

$$\varphi_{p,q} = \frac{2d(\rho_1 - \rho_2)}{4d - \rho_1 - \rho_2}\, k_{p,q} - \left(q + \frac{1}{2}\right) \arcsin\left[(\rho_1 - \rho_2)\sqrt{\frac{2d(\rho_1 + \rho_2 - 2d)}{\rho_1 \rho_2}}\right].$$

In formula (3), in the case of the condition $U|_S = 0$, one should take, for even p, the sine, and for odd p, the cosine (if $\partial U/\partial n|_S = 0$, then the sine and cosine are interchanged).

The eigenfunctions $U_{p,q}$ associated with the system of rays arising in a neighborhood of the minimal diameter oscillate for

$$|\eta| < \eta_{p,q} = \sqrt{\frac{2}{k_{p,q} a}\left(q + \frac{1}{2}\right)}$$

and decrease exponentially for $|\eta| > \eta_{p,q}$.

The conditions of applicability of formula (7) may be written in the form

$$\pi q \sqrt{(2d - \rho_1)(2d - \rho_2)(\rho_1 + \rho_2 - 2d)} \gg \sqrt{d}\, |4d - \rho_1 - \rho_2|,$$

$$|\sqrt{2k_{p,q} a} \sin \eta| < c, \qquad q < c'/4.$$

The proposed method for constructing asymptotic formulas carries over to the case of the Helmholtz equation with a variable coefficient and to the three-dimensional case, provided only that it is possible to construct, stable in the first approximation, a system of rays (solutions of the eikonal equations).

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Note: Figure translations are in progress. See original paper for figures.