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Abstract

Full Text

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ESTIMATES OF THE H-CONDITION NUMBERS OF MATRICES OF THE PETROV-GALERKIN METHOD

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1.

Consider the equation

$$Lx = f, \tag{1}$$

where L is a linear bounded operator from a Hilbert space H_1 into a Hilbert space H_2, having a bounded inverse L^{-1}, x is the unknown element of H_1, and f is a given element of H_2.

Let $\{\varphi_k\}_{k=1}^{\infty}$ be a sequence in H_1 that is complete and linearly independent in any finite number of its elements; let $\{\psi_k\}_{k=1}^{\infty}$ be a sequence of elements of H_2 that are linearly independent in any finite number.

We shall seek an approximate solution of equation (1) by the Petrov-Galerkin method, i.e. in the form

$$x_n = \sum_{k=1}^{n} C_k^{(n)} \varphi_k, \tag{2}$$

and determine the coefficient vector $C^{(n)} = \{C_k^{(n)}\}_{k=1}^{n}$ from the system of linear equations

$$L_n C^{(n)} = f_n, \tag{3}$$

where $L_n = \{(\psi_i, L\varphi_k)\}_{i,k=1}^{n}$; $f_n = \{(\psi_i, f)\}_{i=1}^{n}$.

2.

Let $\{\bar{\varphi}_k\}_{k=1}^{\infty}$ and $\{\bar{\psi}_k\}_{k=1}^{\infty}$ be orthonormal sequences obtained from the sequences $\{\varphi_k\}_{k=1}^{\infty}$ and $\{\psi_k\}_{k=1}^{\infty}$ by the orthogonalization process. Put

$$\bar{L}_n = \{(\bar{\psi}_i, L\bar{\varphi}_k)\}_{i,k=1}^{n}.$$

Theorem. In order that the sequences of approximate solutions x_n, given by the Petrov-Galerkin method (2)–(3), converge to the solutions $x^* = L^{-1}f$ of equations (1) for arbitrary $f \in H_2$, it is necessary and sufficient that there exist a natural number N and a positive number l such that

$$\|\bar{L}_n^{-1}\| \le l \quad \text{for } n \ge N. \tag{4}$$

Here and below the matrix norm is the Euclidean norm.

The proof of sufficiency of condition (4) is carried out according to the usual scheme. The proof of necessity uses the well-known fact of uniform boundedness of the norms of a strongly convergent sequence of linear operators.

3.

In what follows we consider only the case when the Petrov-Galerkin method converges, for fixed operator L and systems $\{\varphi_k\}_{k=1}^{\infty}$ and $\{\psi_k\}_{k=1}^{\infty}$, for arbitrary $f \in H_2$.

Put

$$\Phi_n = \{(\varphi_i, \varphi_k)\}_{i,k=1}^{n}, \qquad \Psi_n = \{(\psi_i, \psi_k)\}_{i,k=1}^{n}.$$

We have, for $n \geqslant N$ (see (1)),

$$\|L_n^{-1}\| \leqslant \|\overline{L}_n^{-1}\| \, \|\Phi_n^{-1}\|^{1/2} \|\Psi_n^{-1}\|^{1/2} \leqslant l \|\Phi_n^{-1}\|^{1/2} \|\Psi_n^{-1}\|^{1/2}. \tag{5}$$

Further,

$$\|L_n\| \leqslant \|\overline{L}_n\| \, \|\Phi_n\|^{1/2} \|\Psi_n\|^{1/2} \leqslant \|L\| \, \|\Phi_n\|^{1/2} \|\Psi_n\|^{1/2}. \tag{6}$$

4. The H-condition number of a nonsingular matrix A (see (2)) is the number $H(A) = \|A\| \|A^{-1}\|$, where $\|A\|$ is the Euclidean norm of the matrix A.

By (5) and (6) we have

$$H(L_n) \leqslant l \|L\| H^{1/2}(\Phi_n) H^{1/2}(\Psi_n). \tag{7}$$

5. A system of elements $\{\varphi_k\}_{k=1}^{\infty}$ of the Hilbert space H_1 is called normal (see (3)) if there exist, independent of n, bounds for the spectrum of the Gram matrix Φ_n of the system $\{\varphi_k\}_{k=1}^{n}$, i.e. $0 < C \leqslant \lambda_k(\Phi_n) \leqslant C < +\infty$ $(k = 1, \dots, n)$.

By inequality (7), the condition numbers of the matrices L_n are uniformly bounded if the systems $\{\varphi_k\}_{k=1}^{\infty}$ and $\{\psi_k\}_{k=1}^{\infty}$ are normal.

The question of the growth of the numbers $H(\Phi_n)$ in the case where the system $\{\varphi_k\}_{k=1}^{\infty}$ is not normal, for certain concrete systems of functions from the space $L^2[a,b]$, is considered in papers ([4],[5]).

6. Let δL_n be a matrix of order n, and let $\|\delta L_n\| < 1/\|L_n^{-1}\|$. Let δf_n be some vector. Then for $n \geqslant N$ there exists a unique solution $\widetilde{C}^{(n)}$ of the equation

$$L_n\widetilde{C}^{(n)} + \delta L_n\widetilde{C}^{(n)} = f_n + \delta f_n,$$

and (see also (6))

$$\frac{\|C^{(n)} - \widetilde{C}^{(n)}\|}{\|C^{(n)}\|} \leqslant \frac{H(L_n)}{1 - \|\delta L_n\|\|L_n^{-1}\|}\left[\frac{\|\delta L_n\|}{\|L_n\|} + \frac{\|\delta f_n\|}{\|f_n\|}\right].$$

Hence, using (7), we obtain

$$\frac{\|C^{(n)} - \widetilde{C}^{(n)}\|}{\|C^{(n)}\|} \leqslant \frac{l\|L\|H^{1/2}(\Phi_n)H^{1/2}(\Psi_n)}{1 - \|\delta L_n\|\|L_n^{-1}\|}\left[\frac{\|\delta L_n\|}{\|L_n\|} + \frac{\|\delta f_n\|}{\|f_n\|}\right]. \tag{8}$$

In the Petrov-Galerkin method, the principal interest is not the vector C^n, but the element x_n obtained by formula (2). For the error in the determination of x_n we obtain, for $n \geqslant N$ and $\|\delta L_n\| < 1/\|L_n^{-1}\|$, an estimate analogous to (8):

$$\frac{\|x_n - \tilde{x}_n\|}{\|x_n\|} \leqslant \frac{l\|L\|H^{1/2}(\Phi_n)H^{1/2}(\Psi_n)}{1 - \|\delta L_n\|\|L_n^{-1}\|}\left[\frac{\|\delta L_n\|}{\|L_n\|} + \frac{1}{H^{1/2}(\Phi_n)}\frac{\|\delta f_n\|}{\|f_n\|}\right], \quad \tilde{x}_n = \sum_{k=1}^{n}\widetilde{C}_k^{(n)}\varphi_k.$$

Thus, the quantity $l\|L\|H^{1/2}(\Phi_n)H^{1/2}(\Psi_n)$ plays a role analogous to that of the H-condition number in the matrix problem. An indirect estimate for the quantity $l\|L\|H^{1/2}(\Phi_n)H^{1/2}(\Psi_n)$ for a given n is furnished by the growth of the numbers $H(\Phi_n)$ and $H(\Psi_n)$ as n grows. If, when solving equation (1) by method (2)–(3), the systems $\{\varphi_k\}_{k=1}^{\infty}$ and $\{\psi_k\}_{k=1}^{\infty}$ are chosen so that the numbers $H(\Phi_n)$ and $H(\Psi_n)$ are bounded or grow slowly, then this usually ensures a small influence of the relative errors in the matrix L_n and in the vector of free terms f_n on the relative error in the approximate solution x_n.

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Note: Figure translations are in progress. See original paper for figures.