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Abstract

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MATHEMATICS

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ON THE APPLICATION OF SPECTRALLY EQUIVALENT OPERATORS TO THE SOLUTION OF DIFFERENCE ANALOGUES OF STRONGLY ELLIPTIC SYSTEMS

(Presented by Academician S. L. Sobolev on 11 XII 1964)

The present work is a continuation and strengthening of works ([1–6]).

1. Let H_h be a finite-dimensional Hilbert space of vector-functions $v = (v_1, v_2, \dots, v_N)$, defined on some set Ω_h of mesh points in the space of variables $x_1, x_2, \dots, x_p$, where h is a certain vector with positive components, determining Ω_h and H_h uniquely. The scalar product in H_h of the functions u_h, v_h will be denoted by $(u, v)_h$. By $|h|$ we shall understand the upper bound of the components of h.

Let B_h, L_h be operators mapping H_h into H_h, and let B_h be linear. We shall say that the positive definite operator B_h is **spectrally equivalent** to L_h and write $B_h \sim L_h$, if there exist such numbers $h_0 > 0, \delta_1 \geqslant \delta_0 > 0, \delta_2 \geqslant 0, \delta_3 > 0$ that

$$\delta_0(B_h(u-v), u-v)_h \leqslant (L_{hu} - L_{hv}, u-v)_h \leqslant \delta_1(B_h(u-v), u-v)_h, \quad (1)$$

$$\delta_2(B_h(u-v), u-v)_h \leqslant (B_h^{-1}(L_{hu} - L_{hv}), L_{hu} - L_{hv})_h \leqslant \delta_3(B_h(u-v), u-v)_h$$

for any $u_h \in H_h,\ v_h \in H_h,\ h \leqslant h_0$. The numbers $\delta_0, \delta_1, \delta_2, \delta_3$ will be called **equivalence estimates**.

Introduce the notation: $H(B_h)$ is the space with scalar product

$$(u, v)_{B_h} \equiv (B_{hu}, v)_h;$$

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A^* and A^+ are the operators adjoint to A, respectively in H_h and $H(B_h)$; $\lambda(A)$ are the eigenvalues of A. Then, if $L_h^* = L_h$ and (1) is fulfilled, then

$$M \equiv B_h^{-1} L_h = M^+$$

and $\lambda(M) \in [\delta_0, \delta_1]$. Therefore it is expedient to replace the equation

$$L_h u_h = f_h \tag{2}$$

by

$$M u_h = B_h^{-1} f_h^* \tag{2'}$$

and to seek u_h, for example, by the iterative method

$$v_h^{n+1} = g_n(v_h^n, \dots, v_h^0, Mv_h^n, \dots, Mv_h^0, B_h^{-1} f_h), \tag{3}$$

where v_h^n is the n-th approximation to u_h; g_n depends on the method.

Let, for the equation

$$\widetilde{B}_h w_h = F_h \tag{4}$$

there exist an iterative method (we shall call it a $T(\widetilde{B})$-method) that makes it possible rather easily to obtain the relation

$$w_h - y_h^m = T_m(w_h - y_h^0), \qquad \|T_m\| < 1, \tag{5}$$

where y_h^0, y_h^m are, respectively, the initial and final approximations in the $T(\widetilde{B})$-method, and T_m is a linear operator. Take

$$B_h = \widetilde{B}_h (E - T_m)^{-1}, \tag{6}$$

where $Ev_h \equiv v_h$. Then the solution of the equation $B_h v_h = F_h$ will be

$$v_h = B_h^{-1} F_h = y_h^m,$$

where y_h^m is the corresponding approximation in the $T(\widetilde{B})$-method for (4)

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* This also applies to the case of a non-self-adjoint and nonlinear operator L_h.

for $y_h^0 = 0$. Therefore all Mv^n and $B_h^{-1} f_h$ in $(2')$ are easily found.

Lemma. If $\widetilde{B}_h \sim L_h$ with estimates $\tilde{\delta}_0, \tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3$ and $T_m = T_m^*$, $\|T_m\| = q < 1$, $T_m \widetilde{B}_h = \widetilde{B}_h T_m$, then

$$B_h \equiv \widetilde{B}_h (E - T_m)^{-1} \sim L_h$$

with estimates

$$\delta_0 = \tilde{\delta}_0 (1-q), \quad \delta_1 = \tilde{\delta}_1 (1+q), \quad \delta_2 = \tilde{\delta}_2 (1-q)^2, \quad \delta_3 = \tilde{\delta}_3 (1+q)^2;$$

if $L_h = L_h^*$, then $\tilde{\delta}_2 \geqslant \tilde{\delta}_0^2,\ \tilde{\delta}_3 \leqslant \tilde{\delta}_1^2$.

We note that the idea of replacing (2) by $(2')$ for the case of differential operators is contained in ([7]); for difference problems, operators B_h of the form (6) were used in ([1–6]).

2. We restrict ourselves to consideration of the method (3) corresponding to the Richardson method for $(2')$:

$$B_h u_h^{n+1} = B_h u_h^n - \gamma_n (L_h u_h^n - f_h). \tag{7}$$

Here u_h^n is the n-th approximation to u_h, the solution of (2); γ_n is a parameter.

Theorem 1. Under condition (1), the solution of (2) exists, and if $\gamma_n = \gamma^0$, where γ^0 corresponds to

$$\inf_{\gamma > 0} \sup_{\substack{\delta_0 \leqslant \lambda \leqslant \delta_1 \\ \delta_2 \leqslant \xi \leqslant \delta_3}} \left| 1 - 2\gamma\lambda + \gamma^2 \xi \right| \equiv \rho < 1,$$

then

$$\|u_h - u_h^{n+1}\|_{B_h}^2 \leqslant \rho \, \|u_h - u_h^n\|_{B_h}^2. \tag{8}$$

Theorem 2. If $B_h \sim L_h,\ L_h = L_h^*$,

$$\gamma_r = \frac{1}{2} \left[\delta_1 + \delta_0 + (\delta_1 - \delta_0) \cos \frac{2r+1}{2n} \pi \right]^{-1}, \quad r = 0, 1, \dots, n,$$

then

$$\|u_h - u_h^{n+1}\|_{B_h} \leqslant \left[T_n \left(\frac{\delta_1 + \delta_0}{\delta_1 - \delta_0} \right) \right]^{-1} \|u_h - u_h^0\|_{B_h},$$

where

$$T_n(\lambda) = \frac{1}{2} \left[(\lambda + \sqrt{\lambda^2 - 1})^n + (\lambda - \sqrt{\lambda^2 - 1})^n \right].$$

Theorem 3. If B_h is defined by (6), then $\{u_h^n\}$, obtained by (7), coincides with $\{v_h^n\}$ for $v_h^0 = u_h^0$, where $\{v_h^n\}$ is constructed according to the rules: 1) from known v_h^n one computes

$$F_h^n = \widetilde{B}_h v^n - \gamma_n (L_h v^n - f_h);$$

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2) y_h^0 in the $T(\widetilde{B})$-method for solving $\widetilde{B}_h v_h = F_h^n$ is taken equal to v_h^n, and iterations are carried out by the $T(\widetilde{B})$-method to obtain (5); 3) $y_h^m = v_h^{n+1}$.

3. Let in $\Omega = \{x = (x_1, x_2, \dots, x_p) : \ 0 < x_s < 1; \ s = 1, 2, \dots, p\}$ one seek the solution of the system of equations

$$\sum_{l=1}^{N} \sum_{\substack{|\alpha|\leqslant m_r \\ |\beta|\leqslant m_l}} (-1)^{|\alpha|} D^\alpha \left(a_{rl}^{\alpha\beta}(x) D^\beta z_l\right) = f_r(x), \quad r = 1, 2, \dots, N, \tag{9}$$

satisfying the boundary conditions

$$D^\alpha z_l\big|_\Gamma = 0, \quad |\alpha| \leqslant m_l - 1, \quad l = 1, 2, \dots, N, \tag{10}$$

where $z(x) \equiv (z_1, z_2, \dots, z_N)$ is the desired solution:

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p); \quad D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_p^{\alpha_p}; \quad D_s = \partial/\partial x_s; \quad |\alpha| = \sum_{s=1}^{p} \alpha_s,$$

$$|\alpha|_0 = \max_s \alpha_s; \quad \beta = (\beta_1, \beta_2, \dots, \beta_p); \quad m_l \geqslant 1; \quad l = 1, 2, \dots, N;$$

$$a_{rl}^{\alpha\beta} = 0, \quad \text{if } |\beta| > m_r; \tag{11}$$

$$\sum_{l,r=1}^{N} \sum_{\substack{|\alpha|=m_r \\ |\beta|=m_l}} a_{rl}^{\alpha\beta} \xi_l^{(\beta)} \xi_r^{(\alpha)} \geqslant \delta \sum_{l=1}^{N} \sum_{|\alpha|=m_l} (\xi_l^{(\alpha)})^2, \quad \delta > 0, \tag{12}$$

for any real N-dimensional vectors

$$\xi^{(\alpha)} = (\xi_1^{(\alpha)}, \xi_2^{(\alpha)}, \dots, \xi_N^{(\alpha)}).$$

On the grid $x_i = ih$, where $h = (h_1, h_2, \dots, h_p)$, $h_s = 1/N_s$ is the step in x_s, $i = (i_1, i_2, \dots, i_p)$, define

$$\Omega_h^r = \{x_i : \ m_r \leqslant i_s \leqslant N_s - m_r, \ s = 1, 2, \dots, p\}.$$

Then the difference analogue of problem (9), (10) will be the problem of finding –

of finding the grid vector-function $u_h = (u_1, u_2, \dots, u_N)$ satisfying the system

$$L_h^r u_h \equiv \sum_{l=1}^{N} \sum_{\substack{|\alpha|\leq m_r \\ |\beta|\leq m_l}} (-1)^{|\alpha|} \bar{\Delta}^\alpha (a_{rl}^{\alpha\beta} \Delta^\beta u_l(x_i)) = f_r(x_i) \quad \text{for } x_i \in \Omega_h^r,$$

$$r = 1, 2, \dots, N, \tag{13}$$

and the boundary conditions

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$$u_l(x_i) = 0, \quad \text{if } x_i \in \bar{\Omega}_h^l, \quad l = 1, 2, \dots, N. \tag{14}$$

Here $\bar{\Delta}_s, \Delta_s$ are respectively the left and right differences with respect to x_s, $\bar{\Delta}^\alpha = \bar{\Delta}_1^{\alpha_1} \bar{\Delta}_2^{\alpha_2} \cdots \bar{\Delta}_p^{\alpha_p}$, $\Delta^\beta = \Delta_1^{\beta_1} \Delta_2^{\beta_2} \cdots \Delta_p^{\beta_p}$.

In the space of grid functions u_h, whose l-th component is defined on Ω_h^l, we introduce the scalar product

$$(u_h, v_h)_h = \sum_{l=1}^{N} h_1 h_2 \cdots h_p \sum_{x_i \in \Omega_h^l} u_l(x_i) v_l(x_i).$$

Along with the operator L_h, defined by formula (13), and $L_h u_h \equiv (L_h^1 u_h, L_h^2 u_h, \dots, L_h^N u_h)$, consider the operators $\tilde{B}_h$, T_m $(m = (M_1, M_2, \dots, M_N))$:

$$\tilde{B}_h u_h \equiv (\tilde{B}_h^1 u_1, \tilde{B}_h^2 u_2, \dots, \tilde{B}_h^N u_N), \quad T_m \equiv (T^1 u_1, T^2 u_2, \dots, T^N u_N); \tag{15}$$

$$\tilde{B}_h^r u_r \equiv (-1)^{m_r} \sum_{s=1}^{p} \bar{\Delta}_s^{m_r} \Delta_s^{m_r} u_r, \quad T^r u_r \equiv \prod_{k=0}^{M_r - 1} (E^r - \tau_k^r (A_{h,k}^r)^{-1} \tilde{B}_h^r) u_r; \tag{16}$$

$$E^r u_r \equiv u_r, \quad A_{h,k}^r = \prod_{s=1}^{p} \left[E^r + \sigma_{s,k}^r \left((-1)^{m_r} \bar{\Delta}_s^{m_r} \Delta_s^{m_r} + \mu_{s,k}^r E^r \right) \right]. \tag{17}$$

The operator T_m is connected with the iterative method $T(\tilde{B})$ for solving the equation $\tilde{B}_h w_h = F_h$ by means of the iterations

$$A_{h,k}^r (y_r^{k+1} - y_r^k) = -\tau_k^r (\tilde{B}_h^r y_r^k - F_r), \quad r = 1, 2, \dots, N, \tag{18}$$

where τ_k^r, $\sigma_{s,k}^r$, $\mu_{s,k}^r$ are iteration parameters. Method (18) was proposed in ([2],[3],[9]) and may be regarded as a natural generalization of the alternating-direction methods ([10],[11]). We shall use the known fact that, applying method (18) with a suitable set of iteration parameters $\tau_k^r, \sigma_{s,k}^r, \mu_{s,k}^r$, $k = 0, 1, \dots, M_r - 1$; $M_r \asymp |\ln h_r|$, $r = 1, 2, \dots, N$, one can obtain $\|T^r\| = q_r < 1$, and hence

$$\|T_m\| = q < 1, \tag{19}$$

where q_r is an arbitrary number in $(0;1)$, with the expenditure of only $\asymp \dfrac{|\ln h \, \ln q'|}{h_1 h_2 \cdots h_p}$ arithmetic operations, $q = \max\limits_r q_r$, $q' = \min\limits_r q_r$.

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Theorem 4. $\tilde{B}_h = \tilde{B}_h^*,\ T_m = T_m^*,\ \tilde{B}_h T_m = T_m \tilde{B}_h,\ \tilde{B}_h > 0.$

Theorem 5. If conditions (11), (12) are fulfilled, $a_{rl}^{\alpha\beta}(x)$ are bounded in $\bar{\Omega}$, and a_{rr}^{00} are sufficiently large*, then $\tilde{B}_h \sim L_h$; if, moreover, $a_{rl}^{\alpha\beta} = a_{lr}^{\beta\alpha}$, then $L_h = L_h^*$.

Theorem 6. If the conditions of Theorem 5 are fulfilled, and u_h^0 is an initial approximation to u_h, the solution of (13), (14), then, carrying out iterations by method (7) with the operator B_h defined by formulas (6), (15)–(17), the estimate

$$\|u_h - u_h^n\|_{\tilde{B}h} \leq \varepsilon \|u_h - u_h^0\|_{\tilde{B}h} \tag{20}$$

can be obtained with the expenditure of only $\asymp |\ln \varepsilon \cdot \ln h| / h_1 h_2 \cdots h_p$ arithmetic operations; $n \asymp |\ln \varepsilon|$.

* If, for example, all $a_{rl}^{\alpha\beta} = 0$ for $1 \leq |\alpha| + |\beta| < m_r + m_l$, then it is sufficient that $a_{rr}^{00} \geq 0$.

4. The proposed method is also successfully applied to other boundary conditions. For brevity, we restrict ourselves to the self-adjoint problem 2:

$$-\sum_{s=1}^{p} D_s(a_s(x) D_s z) + c(x)u = f, \qquad 0 \leq x_s \leq 1, \quad s = 1, 2, \dots, p; \tag{21}$$

$$[-(-1)^{x_s} a_s(x) D_s z + b_s(x) z]_{x_s = 0;\,1} = 0, \qquad s = 1, 2, \dots, p, \tag{22}$$

where a_s, c, b_s are symmetric matrices of order N,

$$a_s \geq \gamma E, \quad b_s \geq bE, \quad \gamma > 0, \quad b \geq 0, \quad c \geq dE, \quad d \geq 0, \quad b^2 + d^2 > 0, \tag{23}$$

E is the identity matrix; γ, b, d are constants.

The difference analogue of problem 2 will be

$$L_h \bar{u} \equiv -\sum_{s=1}^{p} \bar{\Delta}_s (a_s \Delta_s \bar{u}(x_i)) + \bar{c}\bar{u}(x_i) = \bar{f}(x_i) \quad \text{for } x_i \in \Omega_h; \tag{24}$$

$$[-(-1)^{x_s} a_s \Delta_s \bar{u} + \bar{b}_s \bar{u}](x_i) = 0 \quad \text{for } x_i \in \Gamma_s, \qquad s = 1, 2, \dots, p; \tag{25}$$

$$\Omega_h = \{x_i :\ 1 \leq i_s \leq N_s{-}1,\ s = 1, 2, \dots, p,\ N_s = 1/h_s\}; \quad \Gamma_s = \{x_i :\ i_s = 0;\ N_s;\ i_l = 1, 2, \dots, N_l{-}1,\ l \neq s\};$$

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$\bar{\Delta}_s \bar{u} = \Delta_s \bar{u}$ for $x_s < 1$, $\bar{\Delta}_s \bar{u} = \Delta_s \bar{u}$ for $x_s = 1$; $\Gamma_h = \bigcup \Gamma_s$.

In the space of u_h defined on Ω_h, we define the operator L_h by formula (24), where $\bar{u} = u_h$ for $x_i \in \Omega_h$, and on Γ_h it is extended by the conditions (25); the operator $\widetilde{B}_h$ is specified by the formula

$$\widetilde{B}_h u_h \equiv -(\Delta_h \tilde{u}_1, \Delta_h \tilde{u}_2, \dots, \Delta_h \tilde{u}_N) + d u_h, \tag{26}$$

where $\Delta_h \tilde{u}_r \equiv \sum_{s=1}^{p} \bar{\Delta}_s \Delta_s \tilde{u}_r$; $\tilde{u} = u_h$ for $x_i \in \Omega_h$, and on Γ_h $\tilde{u}$ is extended from the conditions

$$[-(-1)^{x_s} \Delta_s \tilde{u} + \tilde{b}\, \tilde{u}](x_i) = 0 \quad \text{for } x_i \in \Gamma_s, \qquad s = 1, 2, \dots, p.$$

Theorem 7. Suppose (23) is satisfied; a_s, b_s, c are bounded in $\bar{\Omega}$. Then

$$L_h = L_h^*, \qquad \widetilde{B}_h = \widetilde{B}_h^*, \qquad \widetilde{B}_h \sim L_h,$$

where $L_h, \widetilde{B}_h$ are defined by (24)–(26).

Define the operator T_m by (15), (16), understanding by $A_{h,k}^r u = A_{n,k}^r u$,

$$A_{h,k}^r = A_{h,k} \equiv \prod_{s=1}^{p} \left(E - \sigma_{s,k}(\bar{\Delta}_s \Delta_s - \mu_{s,k} E)\right).$$

If the conditions of Theorem 7 are satisfied, then the assertions of Theorems 4 and 6 also hold, with $L_h, \widetilde{B}_h, T_m, B_h$ understood in them as the operators indicated in item 4 and with u_h regarded as the solution of (24), (25).

5. It is useful to bear in mind that if the computation by (18) proceeds with a cyclic repetition of τ, σ, μ leading in one cycle to the estimate $\|T^r\| = \tilde{q}_r < 1$, where $\tilde{q}_r$ does not depend on h, then such a transition from one iteration cycle to a new cycle is equivalent to applying (7) with $L_h = \widetilde{B}_h$, $\gamma_n = 1$, $B_h = (E - T_m)^{-1} \widetilde{B}_h$, $B_h \sim \widetilde{B}_h$.

Remarks. 1) Theorems 5 and 7 have also been obtained for more general domains Ω. 2) Nonhomogeneous boundary conditions (14), (22) do not cause any essential changes in the indicated method. 3) The results of the work, as is clear from Theorem 1, admit generalization also to the case of some nonlinear strongly elliptic systems and systems of integro-differential equations.

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