ON PARABOLIC SYSTEMS WITH COEFFICIENTS SATISFYING THE DINI CONDITION

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Abstract

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ON PARABOLIC SYSTEMS WITH COEFFICIENTS SATISFYING THE DINI CONDITION

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Here we shall set forth results on the correct solvability of the Cauchy problem and the existence of fundamental solutions for linear parabolic systems whose coefficients have, with respect to the spatial coordinates, a modulus of continuity satisfying the Dini condition. Examples ([2]) show that such an assumption on the smoothness of the coefficients is minimal for the validity of the results formulated below. Throughout, the definitions and notation of ([1]) are used.

1. Correct solvability of the Cauchy problem. Consider the Cauchy problem for a system parabolic in the sense of Petrovskii

$$\frac{\partial u}{\partial t} = \sum_{|k|\le 2b} A_k(t,x) D^k u + f(t,x), \tag{1}$$

$$u\big|_{t=+t_0} = \varphi(x). \tag{2}$$

Definition 1. A continuous function $f(t,x)$, bounded in the cylinder

$$Q = [t_0, T] \times \Omega$$

(Ω is a certain domain of n-dimensional space E_n), satisfies the Dini condition with respect to x (belongs to the class H_1) if its modulus of continuity with respect to x

$$\omega(h) = \sup_{|x-\xi|\le h} |f(t,x) - f(t,\xi)|$$

has the property:

$$F(z) = \int_0^z \frac{\omega(h)}{h}\, dh < +\infty$$

for some positive z. If, moreover,

$$F_1(a) = \int_0^a \frac{F(z)}{z}\, dz < +\infty$$

for some positive a, then $f(t,x)$ belongs to the class H_2.

Definition 2. A function $u(t,x)$, defined in the layer $\Pi(t_0,t_1]$, belongs to the class $C^{(m)}_{K(t),N}$ if: 1) $D^k u(t,x)$, $|k| \le m$, belong, for $t > t_0$, to $C_{K(t),N}$ ([1]); 2)

$$\int_{t_0}^{t} \|D^k u(t,x)\|_{C_{k(t),N}} \, dt < +\infty, \qquad |k| \le m;$$

3) $D^k u(t,x)$, $|k| \le m$, are continuous for $t > t_0$, and for each layer $\Pi[t_0^*, t_1]$, $t_0^* > t_0$, there exists a constant C such that

$$|D^k u(t,x) - D^k u(t,y)| \le$$
$$\le C\omega(|x-y|) \left[\exp\left\{ k(t) \sum_s x_s^q \right\} + \exp\left\{ k(t) \sum_s y_s^q \right\} \right].$$

For $|k| = 0$ the last inequality is valid with $t_0^* = t_0$. The class $C^{(m,\omega)}_{a,N}$ of functions depending only on x is defined analogously.

Theorem 1. *Suppose that: 1) $A_k(t,x)$ belong to the class H_1 in the layer $\Pi[t_0,T]$ with the function $\omega(h)$; 2) $A_k(t,x)$ with $|k| = 2b$ are continuous in t, uniformly with respect to x.*

Then, for any $\varphi(x) \in C^{(0,\omega)}_{a,N}$, $f(t,x) \in C^{(0,\omega)}_{K(t),N}$, there exists a unique solution of the Cauchy problem (1), (2), $u(t,x)$, belonging to

$$C^{(2b,F)}_{K(t),N}, \qquad F(z) = \int_0^z \frac{\omega(h)}{h} \, dh.$$

The proof of this theorem is carried out with the aid of E. Hopf' s method ([2]), whose application to the study of parabolic systems with Hölder coefficients is presented in ([1]).

2. Fundamental solution matrices (f.s.m.). Analysis of the formulas by means of which the solution of the Cauchy problem is defined makes it possible to construct the f.s.m. of the Cauchy problem under assumptions 1), 2) of Theorem 1.

Definition 3. We shall call a fundamental solution matrix of the Cauchy problem for system (1) a matrix $Z(t,\tau,x,\xi)$ such that, for any vector-function $\varphi(x) \in C^{(0,\omega)}_{a,N}$,

$$u(t,x) = \int Z(t,t_0,x,\xi)\varphi(\xi) \, d\xi \tag{3}$$

defines a solution of the Cauchy problem 1), 2) ($f = 0$), belonging to $C^{(2b,F)}_{K(t),N}$.

Theorem 2. I. *If the conditions 1), 2) of Theorem 1 are satisfied, then system (1) has a f.s.m. of the Cauchy problem $Z(t,\tau,x,\xi)$, having derivatives with respect to x up to order $2b-1$, for which the estimates*

$$|D^k Z(t,\tau,x,\xi)| \le C(t-\tau)^{-(n+k_1)/2b} \exp\{-c\rho(t,\tau,x,\xi)\} \tag{4}$$

hold, and such that $u(t,x)$, defined by formula (3), has derivatives up to the order of the equation, computed by the formulas

$$D_x^k u(t,x) = \Big[\int \bar{D}_x^k G(t,t_0,x-\xi,x)[\varphi(\xi)-\varphi(x)]\,d\xi +$$

$$+\int_{t_0}^{t} d\beta \int \bar{D}_x^k \widetilde{G}(t,\beta,x-\xi,x) \sum_{|m|\le 2b} \{A_m(\beta,\xi) -$$

$$-A_m(\beta,x)\delta_{|m|,2b}\}\Phi_m(\beta,\xi)\,d\xi\Big]\, e^{A(t-t_0)}, \tag{5}$$

where $\widetilde{G}(t,\tau,x-\xi,x)$ is the Green matrix of the system

$$\frac{\partial u}{\partial t} = \sum_{|k|=2b} A_k(t,y) D_x^k u - Au;$$

A is a certain constant; $\bar{D}_x^k$ denotes differentiation with respect to x of $G(t,\tau,x-\xi,x)$ in the third argument, and $\Phi_k(t,x)$ is the solution of the integral equation

$$\Phi_k(t,x) = \int \bar{D}_x^k G(t,t_0,x-\xi,x)[\varphi(\xi)-\varphi(x)]\,d\xi +$$

$$+\int_{t_0}^{t} d\beta \int \Big[D_x^k G(t,\beta,x-\xi,x) \sum_{|m|\le 2b} \{A_m(\beta,\xi) - A_m(\beta,x)\}\Phi_m(\beta,\xi) +$$

$$+\bar{D}_x^{k-1} G(t,\beta,x-\xi,x) \sum_{|m|\le 2b-1} A_m(\beta,x)\Phi_{m+1}(\beta,\xi)\Big]\,d\xi. \tag{6}$$

II. *If, in addition to conditions 1), 2) of Theorem 1, the following condition is satisfied: 3) the coefficients $A_k(t,x)$ with $|k| = 2b$ belong to H_2, then for the sys-*

there exists an f.m.s. $Z(t,\tau,x,\xi)$ having derivatives of all orders entering (1), for which the estimates

$$\left|D_x^k Z(t,\tau,x,\xi)\right| \leq C(t-\tau)^{-(n+|k|)/2b}\exp\{-c\rho(t,\tau,x,\xi)\}, \quad |k|\leq 2b; \qquad (7)$$

$$|\Delta_h D_x^m Z(t,\tau,x,\xi)| \leq C\,\frac{F(|h|)}{F\left((t-\tau)^{1/2b}\right)}(t-\tau)^{-(n+2b)/2b} \times \left[\exp\{-c\rho(t,\tau,x,\xi)\}+\exp\{-c\rho(t,\tau,x+h,\xi)\}\right], \quad |m|=2b; \qquad (8)$$

C, c are positive constants depending on $\sup|A_k(t,x)|$, the functions $\omega(h)$, $F(h)$, $F_1(h)$, the character of continuity of $A_k(t,x)$ with $|k|=2b$ in t, and the numbers $T, 2b, n, \delta$; δ is the parabolicity constant.

We note that for equations with Hölder coefficients $\omega(h)=Lh^\alpha$ and $F(h)=\dfrac{L}{d}h^\alpha$; therefore it follows from estimate (8) that the higher derivatives of the f.m.s. satisfy the Hölder condition with the same exponent as the coefficients. This result was obtained earlier by S. D. Ivasišen.

It should be emphasized that the f.m.s. which are discussed in the first part of Theorem 2 possess all the properties needed for subsequent applications; in particular, the volume and surface potentials whose kernels are these f.m.s. and their derivatives have the same properties as the potentials constructed by means of the f.m.s. of systems whose coefficients belong to H_2.

The proof of the first part of Theorem 2 is carried out, as already mentioned, by means of E. Hopf' s device, and the second part by the usual Levi method; in doing so the following two lemmas, which, it seems to us, are of independent interest, are used essentially.

Lemma 1. Let $G(t,\tau,x-\xi,y)$ be the Green matrix of the system

$$\frac{\partial u}{\partial t}=\sum_{|k|=2b}A_k(t,y)D_x^k u;$$

$A_k(t,y)$ satisfies conditions 1), 2) of Theorem 1, and $f(t,x)$ the conditions:

1) it is defined and continuous in $\Pi(t_0,T]$;

2)

$$\|f(t,x)\|_{c_{k(t),N}} \leq \frac{C}{b_1(t-t_0)}; \quad |\Delta_h f(t,x)| \leq \frac{\tilde\omega(|h|)}{b_2(t-t_0)}\left[\exp\{k(t)\Sigma x_s^q\}+\exp\{k(t)\Sigma|x_s+h_s|^q\}\right],$$

$\tilde{\omega}(z)$ is a function possessing the properties of a modulus of continuity; $b_i(t)$, $i = 1, 2$, are monotonically nonincreasing functions,

$$b_i\left(\frac{t}{2}\right) \geq K b_i(t); \quad \int_0^a \frac{d\tau}{b_1(\tau)} < +\infty.$$

Then for the derivatives of order $2b$ of

$$u(t,x) = \int_{t_0}^t d\tau \int G(t,\tau,x-\xi,\xi) f(\tau,\xi) d\xi$$

the estimate

$$\left|\Delta_h D^k u(t,x)\right| \leq C \frac{[F(|h|) + \widetilde{F}(|h|)]\,[\exp\{k(t)\Sigma x_s^q\} + \exp\{k(t)\Sigma |x_s + h_s|^q\}]}{\min(b_2(t-t_0),\, t-t_0)\,[F((t-t_0)^{1/2b}) + \widetilde{F}((t-t_0)^{1/2b})]}.$$

is valid. An analogous assertion is valid for derivatives of lower order, for the derivative with respect to t, and for potentials whose kernels are $G(t,\tau,x-\xi,y)$, $Z(t,\tau,x,\xi)$.

Denote by $d(t,\tau,x,\xi) = \sqrt{(t-\tau)^{1/b} + |x-\xi|^2}$ the parabolic distance, and

$$\rho(t,\tau,x,\xi) = \left[d(t,\tau,x,\xi)(t-\tau)^{-1/2b}\right]^q, \quad q = \frac{2b}{2b-1}.$$

Lemma 2. For the integral

$$I(t,\tau,x,\xi;A) = \int_\tau^t d\beta \int e^{-\rho(t,\tau,x,\xi) - A(t-\beta)}$$

$$\times [d(t,\beta,x,y) d(\beta,\tau,y,\xi)]^{-n-2b}\, \omega(d(\beta,\tau,y,\xi))\, dy$$

the estimate is valid

$$I(t,\tau,x,\xi;A) \leqslant \left[C_1 \int_0^\varepsilon \frac{\omega(h)}{h}\, dh + C_2 \omega(\varepsilon) \varepsilon^{-n-2b} A^{-1/2b}\right] \times$$

$$\times [d(t,\tau,x,\xi)]^{-n-2b}\, \omega\big(d(t,\tau,x,\xi)\big);$$

ε is any number in $(0,1)$; C_1, C_2 depend only on $n, 2b$.

In order to illustrate the sharpness of the results obtained, we note that Theorem 1 and the first part of Theorem 2 are valid, in particular, if the modulus of continuity with respect to x of the coefficients is

$$\omega(h) = \left(\ln \frac{1}{h}\right)^{-1-\varepsilon},$$

$\varepsilon > 0$, while at the same time there exist parabolic equations with modulus of continuity

$$\omega(h) = \left(\ln \frac{1}{h}\right)^{-1+\varepsilon},$$

for which the solution of the Cauchy problem can no longer satisfy the inequality

$$|u(t,x)| \leqslant \Psi(t-t_0) \int_{-\infty}^{\infty} |\varphi(\xi)|\, d\xi$$

and, consequently, for such equations the assertions of Theorems 1 and 2 are no longer valid*.

3. Generalizations. All the results set forth extend to $2b$-parabolic systems (4) and to systems of higher order in t.

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CITED LITERATURE

[1] S. D. Eidelman, *Parabolic Systems*, Moscow, 1964.
[2] A. M. Il’ in, DAN, **147**, No. 4 (1962).
[3] E. Hopf, Math. Zs., **34**, 194 (1932).
[4] S. D. Eidelman, DAN, **133**, No. 1, 40 (1960).
[5] M. I. Matichuk, S. D. Eidelman, Dokl. AN UkrSSR, No. 1, 18 (1963).
[6] M. I. Matichuk, DAN, **150**, No. 3, 480 (1963); Dokl. AN UkrSSR, No. 8, 1010 (1964).

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* Examples of such equations were kindly communicated to us by A. M. Il’ in; they are constructed analogously to the examples given by him in [2].

Note: Figure translations are in progress. See original paper for figures.