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Abstract

Full Text

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ON THE ASYMPTOTIC BEHAVIOR AS $t \to \infty$ OF SOLUTIONS OF CERTAIN NONSTATIONARY BOUNDARY-VALUE PROBLEMS

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Consider in the $(n+1)$-dimensional space $(t, x) = (t, x_1, \dots, x_n)$ an n-dimensional cylindrical surface Γ with generator parallel to the axis Ot and with directrix an $(n-1)$-dimensional, closed, N-times continuously differentiable surface S, $\Gamma = S \times (-\infty < t < \infty)$.

Let V_0 be the domain of the plane $t = 0$ bounded by the surface S and containing infinity, and

$$\Omega = V_0 \times [0 \le t < \infty),$$

$$V_\tau = \Omega \cap (t = \tau), \quad \Omega_\tau = \Omega \cap (0 < t < \tau), \quad 0 \le \tau < \infty.$$

Denote by $u(x, t)$ a solution in Ω of the equation

$$\partial^2 u/\partial t^2 + (-\Delta)^m u = f(x) e^{i\omega t}, \tag{1}$$

satisfying the initial conditions

$$u|_{t=0} = a(x); \tag{2}$$

$$\partial u/\partial t|_{t=0} = b(x) \tag{3}$$

and the boundary conditions

$$B_i(\partial/\partial x) u|_\Gamma = 0, \qquad i = 1, \dots, m, \tag{4}$$

where the functions $f(x)$, $a(x)$, and $b(x)$ are assumed to be finite and N-times continuously differentiable functions of their arguments; m is some integer, $m \ge 1$; ω is some real number; the operators $B_i(\partial/\partial x)$, $i = 1, \dots, m$, prescribed on the boundary Γ, are assumed to be linearly independent and to have orders $\mu_i < 2m$, $i = 1, \dots, m$. In addition, we shall assume symmetric the operator

Δ^m, defined on functions in V_0 that are $2m$ times continuously differentiable and satisfy the boundary conditions (4). The function $u(x,t)$, $(x,t) \in \Omega$, will be called a generalized solution of problem (1), (2), (3), (4) if $u(x,t) \in W^{1,m}_{2,t,x}(\Omega_T)$ for any $T > 0$ and if, for the function $u(x,t)$, the integral identity

$$\int_{\Omega_T} u \left(\frac{\partial^2 \bar{v}_T}{\partial t^2} + (-\Delta)^m \bar{v}_T \right) dx\, dt = \int_{\Omega_T} f(x) e^{i\omega t} \bar{v}_T(x,t)\, dx\, dt +$$

$$+ \int_{V_0} b(x) \bar{v}_T(x,0)\, dx - \int_{V_0} a(x) \frac{\partial \bar{v}_T(x,0)}{\partial t}\, dx \tag{5}$$

holds for any function $v_T(x,t) \in W^{2,2m}_{2,t,x}(\Omega_T)$ that vanishes for $t = T$ together with its derivative with respect to t and satisfies on Γ the conditions (4).

Lemma 1. There exists a unique generalized solution of the problem (1), (2), (3), (4).

We shall denote equation (1) for $f(x) \equiv 0$ by (1_0), and condition (2) for $a(x) \equiv 0$ by (2_0).

The solution $u(x,t)$ of the problem (1), (2), (3), (4) can be represented in the form

$$u(x,t) = \frac{\partial u_a(x,t,0)}{\partial t} + \int_0^t u_{fe^{i\omega\tau}}(x,t,\tau)\, d\tau, \tag{6}$$

where $u_b(x,t,\tau)$ is the solution in the half-space $t > \tau$ of equation (1_0), satisfying for $t = \tau$ the conditions (2_0) and (3), and for $t \geq \tau$ the condition (4). This representation makes it possible to prove the following lemma.

Lemma 2. If the functions $a(x)$, $b(x)$, $f(x)$ are sufficiently smooth, then the generalized solution of the problem (1), (2), (3), (4) is a sufficiently smooth function satisfying, in the classical sense, the conditions (1), (2), (3), and (4). Moreover, for the solution of the problem (1_0), (2), (3), (4) the inequality

$$\int_{V_t} \left(\left| \frac{\partial^s u}{\partial t^s} \right|^2 + \sum_{|\alpha| = ms} |D_x^\alpha u|^2 \right) dx \leq C_s \tag{7}$$

holds for $s = 1, \ldots, S$, where S is a sufficiently large number; C_s are certain constants independent of t.

Lemma 3. The solution $u(x,t)$ of the problem (1), (2), (3), (4) is bounded in Ω, together with all its derivatives of sufficiently high order.

This lemma follows immediately from Lemma 2 and Lemma 4.

Lemma 4. Let $w(x)$ be a function of the n variables $x = (x_1, \dots, x_n)$, defined in the whole space R_n and belonging to $W_2^{(n)}(R_n)$. Then

$$\operatorname*{vrai\,max}_{x \in R_n} |w(x)| \leq C \|w\|_{W_2^{(n)}(R_n)}$$

with a constant C independent of the function $w(x)$.

The principal role in the subsequent considerations will be played by the following lemma.

Lemma 5. Let

$$A = \sup_{x \in S} |x|.$$

Then for $|x| \geq 2A$ the inequality

$$|u(x,t)| + |\partial u/\partial t| + |\nabla_x u| + \dots + |\nabla_x^{2m-1} u| \leq Ct^{2\rho(m+2)/m(m+1)}/|x|^{2\rho/(m+1)}, \quad (8)$$

holds, where ∇_x is the gradient with respect to $x_1, \dots, x_n$; ρ is any integer not exceeding N (N is the assumed smoothness of the right-hand sides in (1), (2), (3)).

We shall prove Lemma 5 for $u(x,t)$ which is a solution of the problem (1_0), (2_0), (3), (4). The general result follows from this with the aid of (6).

Let $2m > n$. Denote by $U(x-\xi, t-\tau)$ the fundamental solution of equation (1) in the whole space. Multiplying equation (1_0) by $U(x-\xi, t-\tau)$ and integrating the resulting equality over the domain Ω_t, we obtain

$$u(x,t) = \int_{V_0} b(\xi) U(x-\xi, t)\, d\xi +$$

$$+ \sum_{i=1}^{m} \int_0^t d\tau \int_S B_i\left(\frac{\partial}{\partial \xi}\right) U(x-\xi, t-\tau) g_i(\xi, \tau)\, d\xi, \quad (9)$$

where $g_i(x,t) = A_i(\partial/\partial x) u(x,t)$; $A_i(\partial/\partial x)$ are certain differential operators on Γ, whose orders do not exceed $2m-1$. In any case, by Lemmas 4 and 5, $g_i(x,t)$, $i = 1, \dots, m$, are sufficiently smooth functions on Γ, bounded together with their derivatives. From (9) one can obtain the estimate (8), if one uses the fact that, for any $s = 1, \dots, n$, the following representation of the function $U(x-\xi, t-\tau)$ holds:

$$U(x-\xi,t-\tau)=\frac{1}{(2\pi)^n}\left[\int_{|\alpha'|\geqslant 1}d\alpha'\int_{-\infty}^{+\infty}e^{i(x-\xi,\alpha)}\frac{\sin|\alpha|^m(t-\tau)}{|\alpha|^m}\,d\alpha_s+\int_{|\alpha'|\leqslant 1}d\alpha'\int_{l_2}e^{i(x-\xi,\alpha)}\frac{\sin|\alpha|^m(t-\tau)}{|\alpha|^m}\,d\alpha_s\right]$$

where $\alpha'=(\alpha_1,\dots,\alpha_{s-1},\alpha_{s+1},\dots,\alpha_n)$; l_2 is a contour in the complex α_s-plane, consisting of the intervals of the real axis $(-\infty,-2)$, $(+2,+\infty)$ and the semicircle $|\alpha_s|=2$, $\operatorname{Im}\alpha_s\geqslant 0$ for $x_s-\xi_s\geqslant 0$, and the semicircle $|\alpha_s|=2$, $\operatorname{Im}\alpha_s\leqslant 0$ for $x_s-\xi_s\leqslant 0$.

If $2m\leqslant n$, then one should first obtain, by the method described above, an estimate analogous to (8) for sufficiently high derivatives of $u(x,t)$, and then, by standard devices, also the inequality (8).

Lemma 6. In the domain V_0 there exists a unique solution of the elliptic equation

$$(-\Delta)^m v-\omega^2 v=f(x), \tag{10}$$

satisfying the conditions (4) on S and the radiation conditions ([1]) at infinity. For the Green function of this problem one has the inequality

$$|G(x,\xi)|\leqslant C/|x-\xi|^{(n-1)/2}$$

with a constant C depending only on the boundary S of the domain V_0.

The purpose of this note is to prove the following theorem.

Theorem. If $u(x,t)$ is a solution of the problem (1), (2), (3), (4), then there exists the limit

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T u(x,t)e^{-i\omega t}\,dt=v(x),$$

where $v(x)$ is the solution of equation (10) discussed in Lemma 6.

This theorem is a weakened limiting-amplitude principle ([2]). For $n=1$ for the second boundary-value problem this theorem was proved in ([3]), and for the first boundary-value problem ($m=1,\ n=2$) in ([4]) (even for some domains with an infinite boundary)∗.

The function

$$w(x,t)=\int_0^t u(x,\tau)e^{-i\omega\tau}\,d\tau$$

for any $t>0$ is a solution of the equation

$$(-\Delta)^m w - \omega^2 w = tf(x) + b(x) - i\omega a(x) - u_t(x,t)e^{-i\omega t} - u(x,t)i\omega e^{-i\omega t}$$
$$\equiv tf(x) + h(x,t),$$

where $h(x,t)$ is a function bounded in Ω and satisfying the inequality

* Proof-correction note. Recently the author became acquainted with the note ([5]), in which similar questions

(8), since $a(x)$ and $b(x)$ are finite functions. According to Lemma 6,

$$\frac{1}{t}w(x,t) = \frac{1}{t}\int_0^t u(x,\tau)e^{-i\omega\tau}\,d\tau = \int_{V_0} G(x,\xi)\left[f(\xi) + \frac{h(\xi,t)}{t}\right]d\xi =$$

$$= v(x) + \frac{1}{t}\int_{V_0} G(x,\xi)h(\xi,t)\,d\xi, \tag{11}$$

where $v(x)$ is the desired solution of equation (10). To complete the proof of the theorem it remains to show that the last term in (11) tends to zero as $t \to \infty$. This is done with the aid of Lemmas 3, 5, and 6. Moreover, if one assumes that the number N is sufficiently large, then one can also give an estimate of this remainder term as $t \to \infty$.

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Note: Figure translations are in progress. See original paper for figures.