



Soviet-era science, translated into English

Geophysics

L. A. Signeva, A. I. Felzenbaum

1965

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196501.63710>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Geophysics

L. A. Signeva, A. I. Felzenbaum

On the Theory of Tides in a Fluid with Friction

(Presented by Academician L. I. Sedov, 3 III 1965)

In calculating tides in sea and ocean basins, we encounter the need to take into account the variability of the current velocity as a function of the vertical coordinate. Of the two main causes responsible for this variability, in the present article we shall take into account turbulent friction in seawater, which is the result of vertical exchange of momentum. We note that this effect is most significant in shallow seas, whereas in deep seas and oceans the other cause—the inhomogeneity of seawater—is more important.

As in the work of G. Sverdrup ⁽¹⁾, we shall take friction in seawater into account by introducing into the equations of horizontal motion terms containing second derivatives of the current velocity with respect to the vertical coordinate. At the same time, we shall consider the general case in which the current velocity, in addition to time and the vertical coordinate, depends on both horizontal coordinates*.

We write the equations of motion in the form

$$\frac{\partial u}{\partial t} - A \frac{\partial^2 u}{\partial z^2} - \Omega v = g \frac{\partial \zeta_0}{\partial x}, \quad \frac{\partial v}{\partial t} - A \frac{\partial^2 v}{\partial z^2} + \Omega u = g \frac{\partial \zeta_0}{\partial y}; \quad (1)$$

$$\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} = \frac{\partial \zeta}{\partial t}, \quad (2)$$

where u, v are the horizontal components of the current velocity; A is the kinematic coefficient of vertical exchange (assumed not to depend on the vertical coordinate z , but it may depend on the horizontal coordinates x, y); g is the acceleration of gravity; Ω is the Coriolis parameter; t is time. The components of the total flux S_x, S_y and the function ζ_0 are determined by the formulas

$$S_x = \int_0^H u \, dz, \quad S_y = \int_0^H v \, dz; \quad (3)$$

$$\zeta = \zeta_0 + Q/g\rho, \quad (4)$$

in which $z = 0$ is the undisturbed sea surface; $z = H$ is the bottom of the basin; ζ is the level; Q is the potential of the tide-generating force; ρ is the density of seawater.

At the sea surface there is no wind, and at the bottom the current velocity becomes zero:

$$A \left(\frac{\partial u}{\partial z} \right)_{z=0} = A \left(\frac{\partial v}{\partial z} \right)_{z=0} = 0; \quad (5)$$

$$(u)_{z=H} = (v)_{z=H} = (w)_{z=H} = 0. \quad (6)$$

In condition (6), w is the vertical component of the current velocity. It does not enter the equations of motion (1), (2), but can be determined from

* Sverdrup restricted himself to the case in which the velocity depends on only one of them.

continuity equation in differential form, after the quantities u, v have been found, by the formula

$$w = \int_z^H \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz, \quad (7)$$

which follows from the continuity equation in differential form (2).

On the shoreline L there is no through-flow and, consequently, the normal component of the total flow is zero:

$$(S_n)_L = S_x \cos \alpha + S_y \cos \beta = 0, \quad (8)$$

where $\cos \alpha, \cos \beta$ are the direction cosines of the normal n .

On the liquid boundary of the basin L' , either the water discharge or the level must be prescribed. In the first case

$$(S_n)_{L'} = S_x \cos \alpha + S_y \cos \beta = f(L'), \quad (9)$$

and in the second

$$(\xi)_{L'} = F(L'), \quad (10)$$

where it is assumed that the function $f(L')$ or $F(L')$ is known.

Since the system of equations (1), (2), (4) is linear, the solution may be sought for each harmonic separately. Introducing the quantity $\sigma_k = 2\pi k/T$ (T is the period) and omitting the index k , for the k -th harmonic we shall have:

$$\begin{aligned} u &= u_1 \cos \sigma t + u_2 \sin \sigma t = \operatorname{Re} [\bar{u}e^{-i\sigma t}], \\ v &= v_1 \cos \sigma t + v_2 \sin \sigma t = \operatorname{Re} [\bar{v}e^{-i\sigma t}]; \end{aligned} \quad (11)$$

$$\begin{aligned} \xi &= \xi_1 \cos \sigma t + \xi_2 \sin \sigma t = \operatorname{Re} [\bar{\xi}e^{-i\sigma t}], \\ \xi_r &= \xi_{10} \cos \sigma t + \xi_{20} \sin \sigma t = \operatorname{Re} [\bar{\xi}_0 e^{-i\sigma t}]; \end{aligned} \quad (12)$$

$$Q = Q_1 \cos \sigma t + Q_2 \sin \sigma t = \operatorname{Re} [\bar{Q}e^{-i\sigma t}], \quad (13)$$

where Re denotes the real part and

$$\bar{u} = u_1 + iu_2, \quad \bar{v} = v_1 + iv_2; \quad (14)$$

$$\bar{\xi} = \xi_1 + i\xi_2, \quad \bar{\xi}_0 = \xi_{10} + i\xi_{20}. \quad (15)$$

Substituting expressions (11), (12), (13) into equations (1), (2), (4), we obtain

$$\begin{aligned} i\sigma\bar{u} + A(\partial^2\bar{u}/\partial z^2) + \Omega\bar{v} &= -g(\partial\bar{\xi}_0/\partial x), \\ i\sigma\bar{v} + A(\partial^2\bar{v}/\partial z^2) - \Omega\bar{u} &= -g(\partial\bar{\xi}_0/\partial y); \end{aligned} \quad (16)$$

$$\partial\bar{S}_x/\partial x + \partial\bar{S}_y/\partial y + i\sigma\bar{\xi} = 0; \quad (17)$$

$$\bar{\xi} = \bar{\xi}_0 + \bar{Q}/g\rho, \quad (18)$$

where

$$\bar{S}_x = \int_0^H \bar{u} dz, \quad \bar{S}_y = \int_0^H \bar{v} dz. \quad (19)$$

Introduce the quantities

$$\vartheta = \bar{u} + i\bar{v}, \quad m = \bar{u} - i\bar{v}; \quad (20)$$

$$\mathcal{G} = -\frac{g}{A} \left(\frac{\partial \bar{\xi}_0}{\partial x} + i \frac{\partial \bar{\xi}_0}{\partial y} \right), \quad G = -\frac{g}{A} \left(\frac{\partial \bar{\xi}_0}{\partial x} - i \frac{\partial \bar{\xi}_0}{\partial y} \right). \quad (21)$$

From equations (16)

$$\frac{\partial^2 \mathbf{v}}{\partial z^2} - \frac{i(\Omega - \sigma)}{A} \mathbf{v} = \mathcal{G}; \quad \frac{\partial^2 m}{\partial z^2} + \frac{i(\Omega + \sigma)}{A} m = G, \quad (22)$$

and from conditions (5), (6)

$$A(\partial \mathbf{v} / \partial z)_{z=0} = 0; \quad A(\partial m / \partial z)_{z=0} = 0; \quad (23)$$

$$(\mathbf{v})_{z=H} = 0; \quad (m)_{z=H} = 0. \quad (24)$$

Integrating equations (22) under conditions (23), (24), we obtain

$$\mathbf{v} = -\frac{\mathcal{G}}{j^2} \left[\frac{\text{ch } jz}{\text{ch } jH} - 1 \right]; \quad m = \frac{G}{q^2} \left[\frac{\text{ch } qz}{\text{ch } qH} - 1 \right], \quad (25)$$

where

$$j^2 = i(\Omega - \sigma)/A; \quad q^2 = -i(\Omega + \sigma)/A. \quad (26)$$

Adding and subtracting the equalities (25), we obtain

$$\begin{aligned} \bar{u} &= -\frac{\mathcal{G}}{2j^2} \left[\frac{\text{ch } jz}{\text{ch } jH} - 1 \right] + \frac{G}{2q^2} \left[\frac{\text{ch } qz}{\text{ch } qH} - 1 \right], \\ \bar{v} &= \frac{\mathcal{G}}{2ij^2} \left[\frac{\text{ch } jz}{\text{ch } jH} - 1 \right] - \frac{G}{2iq^2} \left[\frac{\text{ch } qz}{\text{ch } qH} - 1 \right]. \end{aligned} \quad (27)$$

Integrating expressions (27) with respect to z from the undisturbed sea surface $z = 0$ to the bottom $z = H$, we obtain

$$S_x = r\mathcal{G} + sG, \quad S_y = \frac{1}{i}(r\mathcal{G} - sG), \quad (28)$$

where the notations have been introduced

$$r = \frac{\text{th } jH}{2j^3} - \frac{H}{2j^2}; \quad s = \frac{\text{th } qH}{2q^3} - \frac{H}{2q^2}. \quad (29)$$

Substituting expressions (28) into equation (17) and taking relation (18) into account, we obtain

$$\frac{\partial}{\partial x}(r\mathcal{G} + sG) + \frac{1}{i} \frac{\partial}{\partial y}(r\mathcal{G} - sG) + i\sigma(\bar{\zeta}_0 + \bar{Q}/g\rho) = 0. \quad (30)$$

Substituting expressions (28) into condition (8), we obtain

$$(r\mathcal{G} + sG) \cos \alpha + \frac{1}{i}(r\mathcal{G} - sG) \cos \beta = 0. \quad (31)$$

Let us now represent the functions $f(L')$ and $F(L')$ in the form

$$f(L') = f_1 \cos \sigma t + f_2 \sin \sigma t = R[\bar{f}e^{-i\sigma t}]; \quad (32)$$

$$F(L') = F_1 \cos \sigma t + F_2 \sin \sigma t = R[\bar{F}e^{-i\sigma t}], \quad (33)$$

where

$$\bar{f} = f_1 + if_2; \quad \bar{F} = F_1 + iF_2. \quad (34)$$

Substituting expressions (28) into condition (9) or (10), we obtain

$$(r\mathcal{G} + sG) \cos \alpha + \frac{1}{i}(r\mathcal{G} - sG) \cos \beta = \bar{f}, \quad (35)$$

or

$$(\bar{\zeta}_0)_{L'} = \bar{F} - \bar{Q}/g\rho. \quad (36)$$

We have obtained the fundamental solution of the problem: by integrating equation (30) under the boundary conditions (31) and (35) or (36) by one method or another,

we obtain the values of the function $\bar{\xi}_0$. After this, formulas (27) give the values of \bar{u}, \bar{v} at any horizon. The functions \bar{w} are computed by the formula

$$\bar{w} = \int_z^H (\partial\bar{u}/\partial x + \partial\bar{v}/\partial y) dz, \quad (37)$$

which follows from relation (7). Finally, the function ξ is computed by formula (18).

In passing from the complex form to the real one, instead of one equation (30) we shall have a system of two linear equations containing the functions $\partial^2\xi_{10}/\partial x^2$, $\partial^2\xi_{20}/\partial x^2$, $\partial\xi_{10}/\partial x$, $\partial\xi_{20}/\partial x$, $\partial^2\xi_{10}/\partial y^2$, $\partial^2\xi_{20}/\partial y^2$, $\partial\xi_{10}/\partial y$, $\partial\xi_{20}/\partial y$, ξ_{10} , and ξ_{20} .

This system of equations, for an arbitrary basin contour, is comparatively easily integrated by the grid method with the aid of a high-speed electronic computer.

In passing from the complex form to the real one, u_1, u_2, v_1, v_2 are obtained in the form of linear functions of $\partial\xi_{10}/\partial x$, $\partial\xi_{20}/\partial x$, $\partial\xi_{10}/\partial y$, $\partial\xi_{20}/\partial y$, with coefficients that always contain the parameter

$$a = \sqrt{(\Omega + \sigma)/2A}, \quad (38)$$

as well as the parameter

$$a_1 = \sqrt{(\Omega - \sigma)/2A} \quad (39)$$

in the case when $\Omega > \sigma$, or the parameter

$$a_2 = \sqrt{(\sigma - \Omega)/2A} \quad (40)$$

in the case when $\sigma > \Omega$. The solution is also obtained in the special case when $\sigma = \Omega$.

In conclusion, let us note that computations for a specific basin according to the proposed scheme only slightly exceed in difficulty the calculation of tides by the method of W. Hansen ⁽³⁾, developed for the case when the velocity of the tidal current does not vary in the vertical direction.

State Oceanographic
Institute

Marine Hydrophysical Institute
of the Academy of Sciences of the Ukrainian SSR

Received
1 III 1965

REFERENCES

- ¹ H. V. Sverdrup, *Dynamic of tides on the North Siberion Shelf*, Oslo, 1926.
- ² A. I. Felzenbaum, *Theoretical Foundations and Methods for Calculating Established Sea Currents*, Publishing House of the Academy of Sciences of the USSR, 1960.
- ³ W. Hansen, *Deutsch. Hydrogr. Zs.*, **1** (1952).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.