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## Abstract

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# MATHEMATICS

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# ON THE EQUICOMPOSABILITY OF TWO EQUICOMPLEMENTABLE POLYHEDRA

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If two polyhedra $A$ and $A'$ of equal volumes can be complemented, respectively, by a finite number of congruent polyhedra $P_1$ and $P_1'$, $P_2$ and $P_2'$, $\cdots$, $P_k$ and $P_k'$ (all the polyhedra introduced above have no common interior points pairwise) in such a way that both polyhedra so complemented become congruent, then $A$ and $A'$ are called **equicomplementable**. If $A$ and $A'$ can be cut into a finite number of respectively congruent polyhedra, then $A$ and $A'$ are called **equicomposable**.

**Theorem.** *In any homogeneous $n$-dimensional Riemannian space, any two equicomplementable polyhedra are equicomposable.*

An analogous theorem for the special case when the $n$-dimensional space is Euclidean, the group $G$ contains the full group of parallel translations, and the bodies are polyhedra, was proved for $n = 3$ by Sydler ([1,2]) and for arbitrary $n$ by Hadwiger ([3]). Moreover, Sydler and Hadwiger used very subtle special considerations essentially connected with the fact that the bodies are polyhedra and the space is Euclidean: namely, they used the method of "steps," which had already been applied by Euclid to derive the volume of a tetrahedron, introduced the notion of the cylindricity index of a polyhedron, and applied induction on this index.

The proof in the present paper is based on the most general set-theoretic considerations and, although it is presented in the text only for polyhedra, is valid for bodies under very general assumptions concerning them.* We pass to the proof of the theorem.

Let $F_A$ (Fig. 1a) be the polyhedron to which the white polyhedron $A$ is complemented by the shaded polyhedra $P_1, P_2, \ldots, P_k$ (in Fig. 1 we have taken, for simplicity, $k = 2$), and let $F_{A'}$ be the polyhedron to which the white $A'$ is complemented by the shaded polyhedra $P_1', P_2', \ldots, P_k'$, where $F_A$ is congruent to $F_{A'}$, and each $P_i$ is congruent to the corresponding $P_i'$.

Our aim is to prove the equicomposability of the white polyhedra $A$ and $A'$, equicomplemented by the polyhedra $P_1, P_2, \ldots, P_k$ and, respectively, $P_1', P_2', \ldots, P_k'$ to congruent polyhedra $F_{A'}$ and $F_A$. Let $\alpha$ be a certain fixed

Fig. 1

Figure 1: Fig. 1

motion carrying $F_{A'}$ into $F_A$. Then the polyhedron $A$ is

$$F_A - P_1 - P_2 - \cdots - P_k,$$

and the polyhedron $A'$ is congruent to the polyhedron

$$\bar{A} = F_A - Q_1 - Q_2 - \cdots - Q_k,$$

where $Q_1 = \alpha P_1'$, $Q_2 = \alpha P_2'$, $\cdots$, $Q_k = \alpha P_k'$. The $Q_1$, shown by a dotted line, may consist both of parts of the polyhedron $A$ and of parts of some of the polyhedra $P_i$.

Clearly, we may assume that $2v(P_i) < v(A)$ ($v$ is the volume of a polyhedron; $i = 1, 2, \ldots, k$), for otherwise one may subdivide $P_i'$ by planes into smaller parts, and $P_i$ into the same parts. Now subdivide (if necessary) that region of the polyhedron $Q_1$ which consists of parts of the shaded polyhedra $P_1, \ldots, P_k$ into smaller shaded polyhedra $S_1, \ldots, S_p$ (in Fig. 1a $p = 2$) so that polyhedra equal to them could be placed in $A - Q_1$ (where $A - Q_1$ is that part of $A$ which is not covered by $Q_1$), and so that $S_1, \ldots, S_p$ would have no common interior points. This can be done by virtue of the inequality $2v(P_1) < v(A)$.

Let us place these hatched polyhedra $S_1, \ldots, S_p$ in $A - Q_1$, and cut out the white parts of $A - Q_1$ lying under them and put them into the regions of the polyhedron $Q_1$ that have been freed by them. After this, within the boundaries of $Q_1$ there is a white polyhedron assembled from parts of $A$ (Fig. 1b).

**Fig. 1**

Let us now take out this entire composite white polyhedron, which fills the position of the polyhedron $Q_1$, move to the place of $Q_1$ all the parts of the hatched polyhedron $P_1$ that are in $F_A$, assembling them back into a polyhedron congruent to $P_1$ (these parts in Fig. 1b are $r_1$ and $S_1$), and cut the removed composite white polyhedron, also congruent to $P_1$, in the same way as the hatched polyhedron $P_1$, congruent to it and placed in the position of $Q_1$, has been cut (in Fig. 1a $P_1$ is divided into two hatched polyhedra $r_1$ and $S_1$); and the white polyhedra obtained from this cutting we distribute over the places where the parts of $P_1$ lay before their transfer to the place of $Q_1$ (Fig. 1c).

Next we repeat exactly the same process for $Q_2$. The fact that pieces from $P_2, \ldots, P_k$ may already have been cut out and transferred into $A$ in the first

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* For example, in the plane, by bodies one may understand bounded domains with piecewise-analytic boundaries.

process does not prevent us from assembling, in the same way, all parts of the polyhedron $P_2$ within the boundaries of the polyhedron $Q_2$.

In this second step it is essential that $Q_2$ will not overlap $P_1$, which has been moved to the place of $Q_1$, because $P_2'$ did not overlap $P_1'$, while $Q_1$ and $Q_2$ are obtained from $P_1'$ and $P_2'$ by one and the same motion $a$. We proceed analogously in all subsequent steps, considering the polyhedra $Q_3, \dots, Q_n$. In this process the original white polyhedron $A$ successively passes into new white polyhedra equidecomposable with one another and, finally, into the polyhedron $\overline{A}$, which completes the proof.

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## REFERENCES

1. J. Sydler, *Comm. Math. Helv.*, **16** (1943–1944).

2. V. G. Boltyanskii, *Equicomplementary and Equidecomposable Figures*, Moscow, 1956.

3. H. Hadwiger, *Math. Zs.*, **55**, 3 (1952).

*Note: Figure translations are in progress. See original paper for figures.*

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