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Abstract

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A. E. KOSULIN

ONE-DIMENSIONAL SINGULAR EQUATIONS IN GENERALIZED FUNCTIONS

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Consider the equation

$$a(\tau)f(\tau)+\frac{b(\tau)}{\pi i}\int\limits_{L}\frac{f(t)}{t-\tau}\,dt=g(\tau), \tag{1}$$

where $a(\tau)$, $b(\tau)$, and $g(\tau)$ are known functions belonging to the basic* space Φ; L is a closed sufficiently smooth contour. The question of the existence of solutions of equation (1) belonging to the space Φ was solved (^{1}) under the assumption that $a^2(\tau)-b^2(\tau)\neq 0$ on L. Some results in this direction were obtained (^{2}) also under the assumption that either condition I or condition II is satisfied.

Condition I. The quantity $\Delta_1(\tau)=a(\tau)-b(\tau)$ vanishes at the point $\alpha\in L$, while $\Delta_2(\tau)=a(\tau)+b(\tau)$ vanishes nowhere on L.

Condition II. The quantity $\Delta_2(\tau)$ vanishes at the point $\alpha\in L$, while $\Delta_1(\tau)$ does not vanish on L.

Both in condition I and in condition II it is assumed that $b(\tau)\neq 0$ on L and that α is the only root, of integral multiplicity m, of the corresponding function.

The purpose of the present paper is to investigate the question of the existence of solutions of equation (1) belonging to the space Φ', conjugate to Φ, as well as certain properties of these solutions.

1. Let $\widetilde{\Phi}$ be a countably normed space of functions infinitely differentiable on L, where L is a closed infinitely differentiable contour, and let the system of norms be introduced as follows:

$$\|\varphi\|_p=\max_{t\in L}\{|\varphi(t)|,\ |\varphi'(t)|,\dots,\ |\varphi^{(p)}(t)|\}\quad(p=1,2,\dots).$$

Let, in the singular equation (1), $a(\tau)$, $b(\tau)$, $g(\tau)\in\widetilde{\Phi}$ and $a^2(\tau)-b^2(\tau)\neq 0$ on L. Using the results of (^{1}), one can show that the solutions of equation (1) fall into this same space $\widetilde{\Phi}$; the Cauchy operator maps $\widetilde{\Phi}$ into itself, and in this space

Noether' s theorems are valid. In the case when the coefficients of equation (1) satisfy one of conditions I or II, using the results of (^{2}), one can also show that the solutions of equation (1) belong to the space $\widetilde{\Phi}$.

2. Let us now consider the equation

$$a(\tau)f + \frac{b(\tau)}{\pi i}\int\limits_L \frac{f}{t-\tau}\,dt = 0, \tag{2}$$

where $a(\tau)$ and $b(\tau)$ are functions belonging to a certain countably normed space Φ; L is a closed contour such that the Cauchy operator maps Φ into itself; f is the unknown functional belonging to the space Φ'. In this case, by the expression

$$F = \int\limits_L \frac{f}{t-\tau}\,dt$$

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* By a basic space we shall mean a normed or countably normed space of functions differentiable a sufficiently large number of times.

we shall mean the functional $F \in \Phi'$, defined by the equality

$$(F,\varphi) = \left(f, -\int_L \frac{\varphi(t)}{t-\tau}\,d\bar{t}\right) \qquad (\varphi(t) \in \Phi).$$

Below all solutions of equation (2) belonging to the space Φ will be called classical. Suppose for the time being that $a^2(\tau) - b^2(\tau) \neq 0$ on L and that Noether' s theorems are valid in the space Φ.

Theorem 1. *Equation (2) has only classical solutions belonging to the space* Φ'.

3. Let us now consider equation (1) under the assumption that $g \in \Phi'$ and $a^2(\tau) - b^2(\tau) \neq 0$ on L. Denote by Φ_0 the set of all functions $\varphi_0(t) \in \Phi$ representable in the form

$$\bar{a}(\tau)\varphi(\tau) + \frac{1}{\pi i}\int_L \frac{\bar{b}(t)\varphi(t)}{t-\tau}\,d\bar{t} = \varphi_0(\tau); \qquad \varphi(t),\ \varphi_0(t) \in \Phi. \tag{3}$$

It is easy to show that any element of Φ can be represented in the form

$$\varphi(t) = \sum_{j=1}^{k} \varphi_j(t)(\varphi,\varphi^j) + \hat{\varphi}(t), \qquad \hat{\varphi}(t) \in \Phi_0,$$

where $\varphi^j(t)$ $(j = 1, 2, \dots, k)$ is a complete system of classical solutions of equation (2), and $\varphi_j(t) \in \Phi$ are such that $(\varphi^i, \varphi_j) = \delta_{ij}$ $(i, j = 1, 2, \dots, k)$.

Theorem 2*. *In order that equation (1) be solvable in the space* Φ', *it is necessary and sufficient that the conditions*

$$(g, \widetilde{\varphi}^j) = 0, \qquad (j = 1, 2, \dots, \tilde{k}), \tag{4}$$

be satisfied, where $\widetilde{\varphi}^j(t)$ $(j = 1, 2, \dots, \tilde{k})$ *are solutions of the homogeneous equation adjoint to (2).*

Let $\widetilde{\varphi}(t)$ be a solution of the equation

$$\bar{a}(\tau)\widetilde{\varphi}(\tau) + \frac{1}{\pi i}\int_L \frac{\bar{b}(t)\widetilde{\varphi}(t)}{t - \tau}\, d\bar{t} = \hat{\varphi}(\tau)$$

and let conditions (4) be satisfied; then it is easy to show that the functional f_0, defined by the equality

$$(f_0, \varphi) = (g, \widetilde{\varphi}), \tag{5}$$

is a particular solution of equation (1), while the general solution of equation (1) in the space Φ' has the form

$$f = f_0 + \sum_{j=1}^{k} c_j \varphi^j(t).$$

Let a sequence of functionals $g_n \in \Phi'$ be such that conditions (4) are satisfied for all g_n, and let f_0^n be the elements of the space Φ' corresponding to g_n from (5).

Theorem 3. *If the sequence* g_n *converges strongly (weakly) to* g *in the topology of the space* Φ', *then the sequence* f_0^n *also converges strongly (weakly) to* f_0 *in the topology of the space* Φ'. *If* p *is the order of the functional*** g, *then the order of the functional* f_0 *does not exceed* p.

4. In paper (²) it was shown that both under condition I and under condition II, on the assumption that the given functions belong to the class*** H,

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* For $\chi \geq 0$ (χ is the index of equation (1) in the space Φ), the assertion of the theorem follows easily from T

** For the definition of the order of a functional, see (³).

*** H is the class of functions satisfying on L a Hölder condition with positive exponent.

equation (1) reduces to an equation of Fredholm type

$$f(\tau) + \int_L P(t,\tau) f(t)\, dt = Ag, \tag{6}$$

where the explicit form of the operator A—an unbounded regularizer of equation (1)—was written out in (4).

Suppose that L is a sufficiently smooth contour, and the given functions $a(\tau)$, $b(\tau)$, and $g(\tau)$ have $m+1$ continuous derivatives, where m denotes the multiplicity of the root α for $\Delta_1(\tau)$, and let $g(\tau)$ be such that the conditions

$$(g, \widetilde{\varphi}^j) = 0 \quad (j = 1, 2, \ldots, s); \qquad (Ag, \widetilde{\psi}^i) = 0 \quad (i = 1, 2, \ldots, p), \tag{7}$$

are satisfied, where $\widetilde{\varphi}^j(t)$ $(j = 1, 2, \ldots, s)$ are the classical solutions of the homogeneous equation adjoint to (1), and $\widetilde{\psi}^i(t)$ $(i = 1, 2, \ldots, p)$ are all the solutions of the homogeneous equation adjoint to (6) that do not belong to ${}^*D(A^*)$.

Theorem 4. *In order that equation (1) be solvable in the space Φ, it is necessary and sufficient that conditions (7) be fulfilled.*

5. Let us now consider equation (2), and suppose that the functions $a(\tau), b(\tau) \in \Phi$ satisfy the conditions of the preceding item, and that the unknown functional $f \in \Phi'$. Let Φ_0 be the set of functions $\varphi_0(t) \in \Phi$ representable in the form (3), and let

$$\psi(\tau) + \int_L \widetilde{P}(t,\tau) \psi(t)\, dt = \widetilde{A}\psi_0, \tag{8}$$

$$\psi(\tau) = b(\tau)\overline{\varphi(\tau)}, \qquad \psi_0(\tau) = b(\tau)\overline{\varphi_0(\tau)}$$

be the Fredholm-type equation obtained from (3) by regularization with the corresponding unbounded operator $\widetilde{A}$. Define the functionals $f_i \in \Phi'$ as follows:

$$(\widetilde{A}\varphi, \psi_i) = \overline{(f_i, \varphi)} \qquad (i = 1, 2, \ldots, p),$$

where $\psi_i(t)$ $(i = 1, 2, \ldots, p)$ are solutions of the homogeneous equation adjoint to (8), not belonging to $D(\widetilde{A}^*)$.

Theorem 5. *The general solution of equation (2) in the space Φ' has the form*

$$f = \sum_{i=1}^{p} \alpha_i f_i + \sum_{j=1}^{s} \beta_j \varphi^j(t),$$

$$\varphi^j(t) \quad (j = 1, 2, \dots, s)$$

is a complete system of classical solutions of equation (2); moreover, f_i $(i = 1, 2, \dots, p)$ and $\varphi^j(t)$ $(j = 1, 2, \dots, s)$ are linearly independent.

From this we readily obtain an expression for finding the index $\varkappa$ of equation (2) in the space Φ':

$$\varkappa = \begin{cases} p, & k \geq 0, \\ p - p_1, & -m \leq k \leq 0, \\ -p_1, & k \leq -m; \end{cases} \qquad \varkappa = \begin{cases} p, & k_1 \geq m, \\ p - p_1, & 0 \leq k_1 \leq m, \\ -p_1, & k_1 \leq 0, \end{cases}$$

where the formula on the left applies when $\Delta_1(\alpha) = 0$, while $\Delta_2(\tau) \neq 0$ on L, and the formula on the right when $\Delta_2(\alpha) = 0$, while $\Delta_1(\tau) \neq 0$ on L; p is the number of classical solutions of the homogeneous equation adjoint to (8), and p_1 is the corresponding number for the equation adjoint to (2), and

$$k = -\frac{1}{2\pi i} \ln \frac{\Delta_2(\tau)}{\Delta_1(\tau)(\tau - \alpha)^{-m}} \bigg|_L ; \qquad k_1 = -\frac{1}{2\pi i} \ln \frac{|\Delta_2(\tau)|}{\Delta_1(\tau)(\tau - \alpha)^{m}} \bigg|_L .$$

Theorem 6. If in equation (1) $g \in \Phi'$, and the coefficients satisfy one of conditions I or II, then, for equation (1) to be solvable in the space Φ', it is necessary and sufficient that the conditions

$$(g, \widetilde{\varphi}^j) = 0 \qquad (j = 1, 2, \dots, s')$$

be satisfied, where $\widetilde{\psi}^i(t)$ are the classical solutions of the equation adjoint to (2). In this case the general solution of equation (1) is given by the formula

$$f = f_0 + \sum_{i=1}^{p} \alpha_i f_i + \sum_{j=1}^{s} \beta_j \varphi^j(t).$$

Here the functional f_0 is defined by equality (5).

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* It follows from (4) that such $\widetilde{\psi}^i(t)$ $(i = 1, 2, \dots, p)$ will be found.

In conclusion we note that all the results remain valid if $\Delta_1(\tau)$ or $\Delta_2(\tau)$ vanish not at one point, but at several points belonging to L, provided that their number is finite.

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Leningrad Electrotechnical Institute
named after V. I. Ulyanov (Lenin)

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Note: Figure translations are in progress. See original paper for figures.

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