N. N. CHAUS

1965

SovietRxiv

Abstract

Full Text

N. N. CHAUS

ON CLASSES OF UNIQUENESS OF SOLUTIONS OF THE CAUCHY PROBLEM AND REPRESENTATIONS OF POSITIVE DEFINITE KERNELS

(Presented by Academician S. L. Sobolev on 26 XII 1964)

In this note a general theorem is given on classes of uniqueness of solutions of the Cauchy problem for systems of differential equations with constant coefficients. It is a generalization of the theorem in the author' s note ([1]) and develops the results of S. Täcklind ([2]) and G. N. Zolotarev ([3]) with respect to their extension to general systems with constant coefficients. In the second part of the note an application of similar theorems is given for obtaining multidimensional theorems of Bochner type concerning positive definite (p.d.) kernels. Relying on the general theory developed by Yu. M. Berezanskii in ([4]), we obtain a strengthening of the results of Theorem 4 of ([4]) in the same direction in which the work of E. B. Vul ([5]) strengthens the theorem of B. M. Levitan and N. N. Meiman ([6]).

1°. The following theorem on classes of uniqueness of solutions of the Cauchy problem for systems with constant coefficients is true.

Theorem 1*. *Let the system of differential equations*

$$\frac{\partial u_j(x,t)}{\partial t} = \sum_{k=1}^{m} P_{jk}\left(\frac{\partial}{\partial x}\right) u_k(x,t) \quad (j = 1, \dots, m),$$

where $P_{jk}(\partial/\partial x)$ are polynomials in the derivatives $\partial/\partial x_1, \dots, \partial/\partial x_n$ with constant coefficients, $x = (x_1, \dots, x_n) \in E_n$, have reduced order $p > 1$. Then the totality of all functions $f(x_1, \dots, x_n)$ satisfying the inequality

$$|f(x_1, \dots, x_n)| \le C \exp\left[|x_1|^{p'} l(|x_1|) + \dots + |x_n|^{p'} l(|x_n|)\right] \quad \left(\frac{1}{p} + \frac{1}{p'} = 1\right),$$

where $C = C_f > 0$ is a constant, and $l(s) \in I(p)$, forms a class of uniqueness for the solution of the Cauchy problem for the given system of equations. Here the inclusion $l(s) \in I(p)$ means that $l(s)$, for $s > 0$, is a slowly increasing function ([8]) for which

$$\int_1^{\infty} \frac{ds}{s\, l(s)^{p-1}} = \infty$$

and $s^{p'} l(s)$ is convex.

The proof of this theorem is a development of the proof of the theorem in ([1]).

2°. We pass to applications of 1° to establishing integral representations of p.d. kernels. For simplicity of formulation we restrict ourselves to the scalar case ($m = 1$) and to the case of ordinary differential expressions. Moreover, we shall use not Theorem 1 itself, but a certain variant of it. Namely, let $\mathcal{L}$ be a differential expression in ordinary derivatives of order $r > 1$ with constant coefficients. Consider in the Hilbert space $L_2(\tau\, dx)$, with weight

$$\tau(x) = \exp\left[|x|^{r'} l(|x|)\right] \quad (1/r + 1/r' = 1,\ l(s) \in I(r)),$$

the operator B, which is obtained after the closure of the operator

$$\varphi(x) \to (\mathcal{L}\varphi)(x), \quad \varphi(x) \in C_0^{\infty}(-\infty < x < \infty)$$

in $L_2(\tau\, dx)$.

---

* The author uses the terminology of the book ([7]).

Lemma. In the Hilbert space $L_2(\tau dx)$ there is uniqueness of weak solutions of the equations

$$\frac{du_t}{dt} \pm iB^* u_t = 0 \qquad (0 \le t < \infty).$$

The proof of the lemma may be, with some changes, the proof of Theorem 1 given in ([1]) (in this proof the lemma is essentially the main intermediate result).

2°. Let us turn to the question of the representation of p.d. kernels. A kernel $K(x, y)$ $(x, y \in E_n)$ is called **positive definite** if, for any finite $\varphi(x)$,

$$\iint K(x, y)\varphi(x)\overline{\varphi(y)}\, dx\, dy \ge 0.$$

Let us take a system of differential expressions $\mathcal{L}^{(j)}$ $(j = 1, \dots, n)$ in the ordinary derivatives, of orders $r_j > 1$, with constant coefficients. We shall call p.d. kernels $\Omega_\lambda(x, y)$ $(\lambda = (\lambda_1, \dots, \lambda_n))$ **elementary kernels** if

$$\mathcal{L}^{(j)}_{x_j}\Omega_\lambda(x, y) = \overline{\mathcal{L}}^{(j)}_{y_j}\Omega_\lambda(x, y) = \lambda_j \Omega_\lambda(x, y) \qquad (j = 1, \dots, n). \tag{1}$$

Here $\mathcal{L}^{(j)}_{x_j}$ denotes the action of the expression $\mathcal{L}^{(j)}$ with respect to the variable x_j; the bar denotes passage to the complex-conjugate coefficients. Under certain conditions the kernel K admits the representation

$$K(x,y) = \int_{-\infty}^{\infty} \Omega_\lambda(x,y)\, d\rho(\lambda) \qquad (x, y \in E_n) \tag{2}$$

with a nonnegative finite measure $d\rho(\lambda)$, and the integral converges absolutely. Let us note that $\Omega_\lambda(x,y)$, as solutions of equations (1), can be expressed in terms of a fundamental system of solutions of the equations $\mathcal{L}^{(j)}u = \lambda_j u$, which we shall use in considering an example.

Theorem 2. Suppose that in the n-dimensional space E_n a continuous p.d. kernel $K(x,y)$ $(x, y \in E_n)$ is defined, satisfying, in the sense of generalized functions, the relations

$$\mathcal{L}^{(j)}_{x_j} K(x,y) = \overline{\mathcal{L}}^{(j)}_{y_j} K(x,y) \qquad (j = 1, \dots, n),$$

where $\mathcal{L}^{(1)}, \dots, \mathcal{L}^{(n)}$ are differential expressions with constant coefficients in the variables $x_1, \dots, x_n$, respectively, and of orders $r_1, \dots, r_n$ $(r_j > 1)$. If the estimate

$$|K(x,y)| \le C \exp[|x_1|^{r'_1} l_1(|x_1|) + |y_1|^{r'_1} l_1(|y_1|) + \cdots + |x_n|^{r'_n} l_n(|x_n|) + |y_n|^{r'_n} l_n(|y_n|)],$$

where $1/r_j + 1/r'_j = 1,\ l_j(s) \in I(r_j)$, holds, then there exists a unique representation (2).

We indicate the course of the proof. Introduce

$$\tau_j(x_j) = \exp[|x_j|^{r'_j} 3 l_j(|x_j|)]$$

$$\left(\frac{1}{r_j} + \frac{1}{r'_j} = 1,\ l_j(s) \in I(r_j) \right),$$

so that

$$K \in L_2(\tau_1^{-1}(x_1) \tau_1^{-1}(y_1) \cdots \tau_n^{-1}(x_n) \times \tau_n^{-1}(y_n)\, dx\, dy).$$

According to Theorem 3 of ([4]), our theorem will be true if in each $L_2(\tau_{jdx} j)$, respectively, uniqueness holds for weak solutions of the equations

$$du_t/dt \pm i B_j^* u_t = 0$$

(the operators B_j are constructed here from $\mathcal{L}^{(j)+}$ in exactly the same way as the operator B was previously constructed from $\mathcal{L}$; the plus sign means passage to the adjoint expression). But uniqueness of weak solutions will hold by the lemma.

3°. The conditions of Theorem 1 are, in a certain sense, necessary conditions. Thus, G. N. Zolotarev ([3]) showed that if a system of equations is $2b$-parabolic in the sense of I. G. Petrovskii and with one spatial···

variables, then the convergence of the integral

$$\int_1^\infty \frac{ds}{sl(s)^{2b-1}}$$

implies nonuniqueness of the solution of the corresponding Cauchy problem in the class of functions

$$|f(x)| \le C_f \exp[|x|^{p'} l(|x|)]$$

$(p = 2b,\ 1/p + 1/p' = 1)$. In this connection it is interesting to know how complete the results which, in turn, Theorem 2 can give are. For this purpose we shall consider one well-studied representation. Namely, let $k(t)$ $(-\infty < t < \infty)$ be a continuous, real, even function for which $K(x,y) = \frac{1}{2}(k(x+y) + k(y-x))$ is positive definite. For the kernel $K(x,y)$ one can write the representation (2), taking $\mathcal{L} = -d^2/dx^2 = \mathcal{L}^+$, for which the eigenfunctions are $\cos\sqrt{\lambda}\,x$ and $\sin\sqrt{\lambda}\,x$. Owing to the evenness of $k(t)$, the representation will have the form

$$K(x,y) = \int_{-\infty}^{\infty} \cos\sqrt{\lambda}\,x \cos\sqrt{\lambda}\,y\, d\sigma(\lambda),$$

and, by Theorem 2, $d\sigma(\lambda)$ will be determined by $K(x,y)$ uniquely if

$$|K(x,y)| \le C \exp[x^2 l(|x|) + y^2 l(|y|)]$$

$(C > 0,\ l(s) \in I(2))$. From the last representation it is easy to obtain that

$$k(t) = \int_{-\infty}^{\infty} \cos\sqrt{\lambda}\,t\, d\sigma(\lambda), \tag{3}$$

and for the uniqueness of $d\sigma(\lambda)$ it is sufficient to require

$$|k(t)| \le C_1 \exp[t^2 l(|t|)]$$

$(l(s) \in I(2))$. It turns out that this requirement cannot be weakened. More precisely, if one considers the set of all $k(t)$ which have the representation (3) and are such that

$$|k(t)| \le C_k \exp[t^2 h(|t|)] \quad \left(C_k > 0,\ \int_1^\infty \frac{ds}{sh(s)} = C_0 < \infty\right),$$

then in this set there exists a function $k_0(t)$ admitting the representation (3) with different $d\sigma(\lambda)$.

For the proof, denote by $G(x)$ the Young conjugate function to $H(x) = x^2 h(x)$. We have

$$\int_1^\infty \frac{x\,dx}{H(x)} = C_0 < \infty.$$

Now we use the following inequalities, which hold for any two Young-conjugate functions $G(x)$ and $H(x)$ ([9]):

$$G(t) \le tG'(t) \le G(2t), \quad t \ge 0; \qquad x \le H^{-1}(x)G^{-1}(x) \le 2x, \quad x \ge 0$$

(here $H^{-1}(x)$ and $G^{-1}(x)$ are the inverse functions respectively to $H(x)$ and $G(x)$). Using these inequalities, it is easy to verify the equivalence of the following integrals in the sense of convergence ($a_i > 0$):

$$\int_{a_1}^\infty \frac{G'(t)}{t^2}\,dt, \qquad \int_{a_1}^\infty \frac{dG(t)}{t^2}, \qquad \int_{a_2}^\infty \frac{dx}{[G^{-1}(x)]^2}, \qquad \int_{a_3}^\infty \frac{[H^{-1}(x)]^2}{x^2}\,dx, \qquad \int_{a_4}^\infty \frac{u^2 H'(u)}{H^2(u)}\,du, \qquad \int_{a_5}^\infty \frac{u\,du}{H(u)}$$

Thus we have obtained that, for our function $H(x) = x^2 h(x)$, the integral

$$\int_1^\infty \frac{G'(t)}{t^2}\,dt$$

converges. We shall now use that part of a theorem of E. B. Vul ([5]) which concerns necessity. Namely,

In (5) it is proved that in the class of all functions $f(t)$ $(-\infty < t < \infty)$ representable by absolutely convergent integrals $f(t) = \int_{-\infty}^{\infty} \cos\sqrt{\lambda}\,t\,d\omega(\lambda)$ with complex-valued measures $d\omega(\lambda)$ and for which $\int \exp[|\sqrt{\lambda}|\,|t|]\,|d\omega(\lambda)| \le C\exp[t^2 h(t)]$, there exists an $f_0(t)$ having a non-unique representation if and only if $\int_1^\infty \frac{G'(x)}{x^2}\,dx$ converges. Hence the existence of a function $k_0(t)$ with the required properties follows easily.

Thus, the conditions of Theorem 2 in the case of the expression $\mathcal{L} = -d^2/dx^2$ coincide with the conditions of E. B. Vul. We have so far been unable to prove the necessity of the conditions of Theorem 2 for the uniqueness of the representation (2) in the case of general differential expressions $\mathcal{L}$.

In conclusion, the author expresses gratitude to Yu. M. Berezanskii for his assistance in the work.

Institute of Mathematics
Academy of Sciences of the Ukrainian SSR

Received
23 XII 1964

SovietRxiv

CITED LITERATURE

[1] N. N. Chaus, *Ukrainian Mathematical Journal*, **16**, No. 3 (1964).
[2] S. Täcklind, *Nora Acta Reqial. Soc. Sci. Upsaliensis* (4), **10**, 1936.
[3] G. N. Zolotarev, *Izv. Vyssh. Uchebn. Zaved., Matematika*, 2(3) (1958).
[4] Yu. M. Berezanskii, *DAN*, **136**, No. 5 (1961).
[5] E. B. Vul, *DAN*, **129**, No. 4 (1959).
[6] B. M. Levitan, N. N. Meiman, *DAN*, **81**, No. 5 (1951).
[7] I. M. Gel' fand, G. E. Shilov, *Generalized Functions*, Vol. 3, Moscow, 1958.
[8] B. Ya. Levin, *Distribution of Zeros of Entire Functions*, Moscow, 1956.
[9] M. A. Krasnosel' skii, Ya. B. Rutitskii, *Convex Functions and Orlicz Spaces*, Moscow, 1958.

Note: Figure translations are in progress. See original paper for figures.