POSITIVE DEFINITE FUNCTIONS ON THE GROUP OF MATRICES WITH ELEMENTS FROM A DISCRETE FIELD

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Abstract

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POSITIVE DEFINITE FUNCTIONS ON THE GROUP OF MATRICES WITH ELEMENTS FROM A DISCRETE FIELD

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MATHEMATICS

1. Let $GL(n, K)$ denote the group of all nonsingular matrices of order n with elements from the field K, and $SL(n, K)$ the subgroup of matrices with determinant 1. At present there is a fairly complete theory of unitary representations of these groups in the case when K is the field of complex numbers ([1]). For the field of real numbers, p-adic fields, and finite fields, an equally complete theory has been constructed only for $n = 2$ ([2–4]). In all the cases listed it turned out that the group under consideration belongs to type I (this means, in particular, that every unitary representation of such a group decomposes uniquely into irreducible ones, so that the problem of describing all representations reduces to the description of irreducible representations).

Thus, among all locally compact fields, only the case of an infinite discrete field remained uninvestigated.* For this case it was known only that the groups $GL(n, K)$ and $SL(n, K)$ do not belong to type I. The problem of describing all factor representations (in particular, all irreducible representations) of groups not belonging to type I was until recently considered hopeless. However, in recent works of E. Thoma ([5],[6]) it was shown that for some such groups it is possible to describe all positive definite functions that are constant on conjugacy classes, and thereby to find all factor representations of type II_1 (for the definition of factors of type II_1, see, for example, ([7]), Ch. VII).

In the present work this problem is solved for the groups $GL(n, K)$ and $SL(n, K)$, where K is an infinite discrete field.

The final result may be formulated as follows. Denote by $M(G)$ the totality of all positive definite functions φ on the group G, constant on conjugacy classes and normalized by the condition $\varphi(e) = 1$. Since the set $M(G)$ is convex and compact (in the topology of pointwise convergence), for its description it is sufficient to indicate its extreme points.

Let C be the center of the group G, and $\widehat{C}$ the group of characters of C.

Theorem 1. *If* $G = GL(n, K)$ *or* $SL(n, K)$, *where* K *is an infinite discrete field, then the set* $M(G)$ *has the following extreme points:*

$$1) \quad \varphi_\pi^{(1)}(g) = \begin{cases} \pi(g), & \text{for } g \in C, \\ 0, & \text{for } g \notin C, \end{cases}$$

where π *is an arbitrary element of* $\widehat{C}$*;*

$$2) \quad \varphi_\pi^{(2)}(g) = \pi(\det g),$$

where π *is a multiplicative character of the field* K*.*

Apparently, Theorem 1 is valid for any semisimple Chevalley-Dickson groups over an infinite discrete field K.

From Theorem 1 there follows the following

Theorem 2. *All factor representations of type* II_1 *of the groups* $GL(n, K)$ *and* $SL(n, K)$ *are obtained (up to quasiequivalence) from the decomposition of the regular representation into factors.*

It would be interesting to determine for which class of groups Theorem 2 holds. For example, is it true for algebraic groups over the field K? It follows from results of Thom $(^{5,6})$ that this theorem is valid for the simplest solvable algebraic group—the group of matrices of the form

$$\begin{pmatrix} b & \\ 1 & \end{pmatrix}$$

—and is not valid for the infinite symmetric group.

2. We shall briefly present the proof of Theorem 1. Let $\varphi \in M(G)$. Consider the space H^0 of finite functions on G and introduce in this space the scalar product

$$(f_1, f_2) = \sum_{g_1 \cdot g_2} \varphi(g_1 g_2^{-1}) f(g_1) f(g_2).$$

Denote by H the completion of H^0 with respect to this scalar product. It is easy to verify that the left and right shifts on G preserve the scalar product in H^0 and therefore extend to unitary operators in H. One can also show that the von Neumann algebras L and R, generated respectively by the operators of left and right shifts, are mutual commutants. These algebras are factors if and only if φ is an extreme point of the set $M(G)$. In the general case they decompose

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* Infinite discrete fields differ from all other locally compact fields in that their additive groups are not self-dual in the sense of Pontryagin.

into factors, and this decomposition corresponds to the decomposition of φ into a linear combination (possibly continuous) of extreme points.

We shall consider in detail only the case $G = GL(2, K)$, since for the general case the arguments are quite analogous, though more cumbersome. The group $GL(2, K)$ has the following classes of conjugate elements: a) classes $C(s, \Delta)$ of nonscalar matrices with trace s and determinant Δ; b) classes $C(\lambda)$ of scalar matrices with eigenvalue λ. Denote by $\varphi_1(s, \Delta)$ and $\varphi_2(\lambda)$ the values of the function φ on these classes. Let δ_g denote the function on G equal to 1 at the point g and to zero at all other points. Obviously,

$$(\delta_{g_1}, \delta_{g_2}) = \varphi(g_1 g_2^{-1}).$$

Consider the functions $\delta_{g(x)}$, where $g(x)$ is a matrix of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Since $g(x_1)g(x_2)^{-1} = g(x_1 - x_2)$ and $g(x) \in C(2, 1)$ for $x \neq 0$, we obtain:

$$(\delta_{g(x_1)}, \delta_{g(x_2)}) = \begin{cases} 1 & \text{if } x_1 = x_2, \\ \varphi_1(2,1) & \text{if } x_1 \neq x_2. \end{cases}$$

Simple calculations show that for any sequence $\{x_k\}$ of distinct elements of the field K, the sequence

$$f_k = \frac{1}{k} \sum_{i=1}^{k} \delta_{g(x_i)}$$

is fundamental in H and, consequently, converges to some vector $h \in H$. It is also easily checked that this vector h does not depend on the choice of the sequence $\{x_k\}$ and has the properties

$$(\delta_{g(x)}, h) = (h, h) = \varphi_1(2, 1).$$

Hence the following fact, important for us, follows.

Lemma 1. *If $\{x_k\}$ is a sequence of distinct elements of K, then the sequence $\delta_{g(x_k)}$ converges weakly to some vector h, independent of the choice of x_k.*

Now let g be a matrix of the form

$$\begin{pmatrix} 0 & a \\ -1 & 0 \end{pmatrix}.$$

By Lemma 1, the sequence

$$c_k = (\delta_g, \delta_{g(x_k)})$$

must have a limit. But

$$c_k = \varphi(g \cdot g(x_k)^{-1}) = \varphi_1(x_k, a).$$

Thus the set of values of the function $\varphi_1(x, a)$, for fixed a, has only one limit point.

Now let $g(\mu, x)$ denote the matrix

$$\begin{pmatrix} \mu & x \\ 0 & \mu^{-1} \end{pmatrix}.$$

Then the scalar product

$$(\delta_{g(\mu_1, x_1)}, \delta_{g(\mu_2, x_2)})$$

is equal to 1 for $\mu_1 = \mu_2$, $x_1 = x_2$, while for $\mu_1 \neq \mu_2$ it belongs to some set having only one limit point. Hence it follows that

Lemma 2. *For any sequence of distinct elements* $\{\mu_k\}$ *and any sequence* $\{x_k\}$, *the sequence* $\{\delta_{g(\mu_k, x_k)}\}$ *converges weakly to some vector* h_1, *independent of the choice of* μ_k, x_k.

From Lemma 2 it follows that the sequence $(\delta_g, \delta_{g(\mu_k, x_k)}) = \varphi(g \cdot g(\mu_k, x_k)^{-1})$ has a limit. Choosing as g the matrix $\begin{pmatrix} 0 & a \\ -1 & 0 \end{pmatrix}$, we obtain that the sequence $\varphi_1(x_k, a)$ has a limit for any sequence $\{x_k\}$ of (not necessarily distinct) elements of K. But this is possible only when $\varphi_1(x, a)$ in fact depends only on a: $\varphi_1(x, a) = \psi(a)$. Considering the restriction of the function $\varphi(g)$ to the subgroup of matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, it is not hard to show that $\psi(a)$ is a positive definite function on the multiplicative group K^* of the field K.

Let us now compare $\varphi(g)$ with the positive definite function $\widetilde{\varphi}(g) = \psi(\det g)$. We shall show that the difference $\widetilde{\varphi}(g) - \varphi(g)$ is also a positive definite function. Since this difference is concentrated on the center C of the group G, it is necessary to verify that $\varphi_2(\lambda) - \psi(\lambda^2)$ is a positive definite function on K^*. For this it is enough to take an arbitrary finite function $f(g)$, concentrated on C, and compute the scalar square of the difference $f(g) - f(g \cdot g(x))$. We shall not present these obvious calculations here.

Thus, we have represented an arbitrary function $\varphi \in M(G)$ as the sum of two functions, also belonging to $M(G)$, one of which depends only on $\det g$, while the other is concentrated on C.

Theorem 1 follows directly from this fact and Bochner' s theorem on positive definite functions on the commutative group K^*.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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