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Abstract

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MATHEMATICS

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LOCAL MINIMA OF THE DENSITY OF A LATTICE COVERING OF FOUR-DIMENSIONAL EUCLIDEAN SPACE BY EQUAL SPHERES

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Let a point lattice Γ be given in four-dimensional Euclidean space E^4. As parameters of the lattice we choose the scalar products g_{kl} $(k, l = 1, 2, \ldots, 5;\ k \neq l)$ of the vectors $\mathbf{a}_k$ of a Selling frame ([1]) of this lattice. Denote by $D_4(\Gamma)$ the density of the covering of the space E^4 by spheres of equal radii with centers at the points of the lattice Γ.

The present note is devoted to the problem of finding the local minima of the density $D_4(\Gamma)$ as a function of the parameters g_{kl} of the lattice Γ, and the problem considered is solved completely. Known results directly or indirectly related to the question of local minima of $D_4(\Gamma)$ are contained in papers ([2–6]).

1. Consider the star γ of the simplicial L-decomposition ([1]) of the lattice Γ, i.e., the set of L-simplices having as a common vertex an arbitrarily chosen point O of the lattice Γ. A hyperplane $P = (O, \mathbf{a}_k, \mathbf{a}_l, \mathbf{a}_m)$ passing through the center O of the star γ and the vectors $\mathbf{a}_k, \mathbf{a}_l, \mathbf{a}_m$ of a Selling frame will be called a **hyperplane of symmetrization** of the star γ (and of the lattice Γ) if, with respect to it, the star γ is affinely symmetric, i.e., there exists an affine transformation of the space E^4 preserving the hyperplane P, which carries the star γ into a star $\tilde{\gamma}$ symmetric with respect to the hyperplane P. Everywhere here Selling frames are assumed to be reduced in the sense of Voronoi ([7,8]).

Lattices on which local minima of the density $D_4(\Gamma)$ are attained will be called **extremal lattices**. A necessary condition that a lattice Γ must satisfy in order to be extremal follows from the following theorem:

Theorem 1. *If the star γ of a four-dimensional lattice Γ is not symmetric with respect to any of its hyperplanes of symmetrization, then the lattice Γ is not extremal.*

We indicate the course of the proof of Theorem 1. The density $D_4(\Gamma)$ is equal to

$$D_4(\Gamma) = \frac{\pi^2}{2} \left(\max_{\nu=1,2,\dots,12} R_\nu \right)^4 V^{-1}, \tag{1}$$

where R_ν $(\nu = 1, 2, \dots, 12)$ denotes the radii of the spheres circumscribed about each of the 12 nonhomologous L-simplices of the lattice Γ, and V denotes the volume of the fundamental parallelepiped of the lattice Γ. Consider the set of shear transformations preserving the hyperplane of symmetrization P and the volume V. We assume that each of the transformations considered changes the parameters g_{kl} of the lattice Γ arbitrarily little.

If the star γ is not symmetric with respect to P, then among the indicated shear transformations there is always one such that $\max_{\nu=1,2,\dots,12} R_\nu$ for the varied lattice Γ^* will be smaller than the same quantity for the initial lattice Γ. After this, the assertion of the theorem follows from formula (1).

2. In the space G^{10} of the parameters g_{kl}, the reduction domain G_{I} of lattices of the first type, in which we shall consider the function $D_4(\Gamma)$, is defined by the inequalities ([1],[8])

$$g_{kl} \leqslant 0, \qquad (k, l = 1, 2, \dots, 5;\ k \neq l). \tag{2}$$

The star γ of lattices of the first type is constructed in such a way that any of the hyperplanes $P = (O, \mathbf{a}_k, \mathbf{a}_l, \mathbf{a}_m)$ $(k, l, m = 1, 2, \dots, 5;\ k \neq l,\ l \neq m,\ m \neq k)$ is a hyperplane of symmetrization. A necessary and sufficient condition for the symmetry of the star γ with respect to the hyperplane $(O, \mathbf{a}_{k_1}, \mathbf{a}_{k_2}, \mathbf{a}_{k_3})$ is the fulfillment, for the lattice parameters, of the three equalities:

$$g_{k_i k_4} = g_{k_i k_5} \qquad (i = 1, 2, 3). \tag{3}$$

From Theorem 1 and relations (3) it follows that the only lattice that can be extremal on the set of lattices of the first type is a unique lattice (we consider similar lattices to be identical), namely the so-called principal lattice Γ_1^4 of the first type, for which all parameters g_{kl} are equal to one another. The extremality of the lattice Γ_1^4 was proved by Gametskii ([2],[3]).

Thus, on the set of lattices of the first type the question of local minima of $D_4(\Gamma)$ is completely solved:

Theorem 2. *In the domain G_{I} of lattices of the first type there exists a unique local minimum of the density $D_4(\Gamma)$, attained on the lattice Γ_1^4, and*

$$D_4(\Gamma_1^4) = \frac{2\pi^2}{\sqrt{125}} = 1.7655\dots. \tag{4}$$

3. One of the reduced domains G_{II} of lattices of the second type is specified by the inequalities ([8], p.363)

$$g_{45} \geqslant 0, \qquad g_{ij} \leqslant 0, \qquad g_{45}+g_{i4} \leqslant 0, \qquad g_{45}+g_{i5} \leqslant 0 \qquad (i,j=1,2,3;\ i \neq j). \tag{5}$$

The star γ of lattices of this type has already only 4 hyperplanes of symmetrization: $P_i = (O, \mathbf{a}_i, \mathbf{a}_4, \mathbf{a}_5)$ $(i = 1, 2, 3)$ and $P_4 = (O, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$.

On the basis of Theorem 1 and equalities (3) we conclude that the extremal lattices of the second type can only be those to which, in the space G^{10}, there correspond points lying at the intersection of the domain G_{II} with the three-dimensional plane:

$$g_{14} = g_{15} = g_{24} = g_{25} = g_{34} = g_{35} = g, \qquad g_{12} = g_{13} = g_{23} = h, \qquad g_{45} = t. \tag{6}$$

The intersection of the domain G_{II} with the plane (6) has been investigated ([5]). Combining the result obtained there with ours, we obtain the theorem:

Theorem 3. *In the domain G_{II} of lattices of the second type, the density $D_4(\Gamma)$ has a unique local minimum, which is attained on the lattice Γ_2^4 with parameters g_{kl} satisfying the conditions (6) and such that*

$$g : h : t = -1 : -0.5441\ldots : 0.5001\ldots, \tag{7}$$

where the numbers $x = -h/g$ and $y = -t/g$ constitute one of the solutions of the system

$$xy^2 - 2x^2 - 3xy + 2(x+y) = 0,$$

$$x(9y^2 - 24y + 7) - 14y^2 + 40y - 18 = 0. \tag{8}$$

On the lattice Γ_2^4 the density is equal to:

$$D_4(\Gamma_2^4) = \frac{\pi^2}{2} \cdot 0.3817\ldots = 1.8836\ldots. \tag{9}$$

4. Let us consider the set of lattices of the third type. In the space G^{10}, one of the reduced domains G_{III} of lattices of the third type is defined by means of the inequalities ((8), p. 366)

$$\begin{gathered} g_{45} \geqslant 0, \qquad g_{14} \leqslant 0, \qquad g_{15} \leqslant 0, \qquad g_{23} - g_{45} \geqslant 0, \\ g_{12} + g_{23} - g_{45} \leqslant 0, \qquad g_{13} + g_{23} - g_{45} \leqslant 0, \qquad g_{24} + g_{23} \leqslant 0, \\ g_{25} + g_{23} \leqslant 0, \qquad g_{34} + g_{23} \leqslant 0, \qquad g_{35} + g_{23} \leqslant 0. \end{gathered} \tag{10}$$

For the star γ of lattices of the third type there are two hyperplanes of symmetrization: $P' = (O, a_1, a_2, a_3)$ and $P'' = (O, a_1, a_4, a_5)$. From Theorem 1 and equalities (3) we obtain that the extremal lattices may be those which correspond to points situated at the intersection of the $\widetilde{G}_{\text{III}}$ domain G_{III} with the five-dimensional plane

$$g_{24} = g_{25} = g_{34} = g_{35} = g, \quad g_{12} = g_{13} = h_1, \quad g_{14} = g_{15} = h_2, \quad g_{23} = t, \quad g_{45} = u. \tag{11}$$

Lattices whose parameters satisfy (11) have, in general, 6 different radii R_ν ($\nu = 1, 2, \dots, 12$).

The usual investigation of the function (1) in the domain $\widetilde{G}_{\text{III}}$ gives the following result:

Theorem 4. *On the set of lattices of the third type there exists a unique extremal lattice Γ_3^4, whose parameters g_{kl} are determined by the relations*

$$g_{12} = g_{13} = -g_{23} = \alpha, \qquad g_{14} = g_{15} = -g_{45} = \beta, \tag{12}$$

$$g_{24} = g_{25} = g_{34} = g_{35} = \alpha + \beta, \qquad \alpha : \beta = (1 + \sqrt{13}) : 2, \qquad \alpha < 0, \quad \beta < 0,$$

with

$$D_4(\Gamma_3^4) = \frac{\pi^2}{2} \cdot 0.39085 \dots = 1.9287 \dots . \tag{13}$$

Theorems 2–4 give a complete answer to the question of local minima of density.

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Note: Figure translations are in progress. See original paper for figures.