ON SOLVING THE DIRICHLET PROBLEM BY THE METHOD OF REFINEMENTS USING DIFFERENCES OF HIGHER ORDERS

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1965

SovietRxiv

Abstract

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MATHEMATICS

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ON SOLVING THE DIRICHLET PROBLEM BY THE METHOD OF REFINEMENTS USING DIFFERENCES OF HIGHER ORDERS

(Presented by Academician A. N. Kolmogorov on 24 II 1965)

In the present paper, the estimates of the irregular component of the error in the solution of the Dirichlet problem for the Poisson (Laplace) equation, obtained by the method of refinements using differences of higher orders, are substantially improved in two cases: when the domain has a smooth boundary (see ([1,2])) or is a rectangle (see ([3])). In addition, an example is given establishing the erroneousness of a theorem of V. S. Korolyuk ([4,5]) concerning the method of refinements using differences of higher orders.

1. Consider the boundary-value problem

$$\Delta u = f \quad \text{in } R; \qquad u = \varphi_j \quad \text{on } \Gamma_j, \quad j = 1, 2, 3, 4, \tag{1}$$

where R is the rectangle $\{0 < x < a,\ 0 < y < b\}$; Γ_j are the sides of the rectangle, including the endpoints. Suppose that

$$f \in C_{2m,\lambda}(\overline{R})^*, \qquad \varphi_j \in C_{2m+2,\lambda}(\Gamma_j), \quad j = 1, 2, 3, 4, \tag{2}$$

where $m \geqslant 1,\ 0 < \lambda < 1$, and, in addition,

$$f^{(2p+2q)}_{x^{2p}y^{2q}}(x_j, y_j) = 0, \quad 0 \leqslant p + q \leqslant m, \quad j = 1, 2, 3, 4,$$

$$\varphi_j^{(2k)}(s_{j1}) = \varphi_j^{(2k)}(s_{j2}) = 0, \quad 0 \leqslant k \leqslant m + 1, \quad j = 1, 2, 3, 4, \tag{3}$$

where (x_j, y_j) are the coordinates of the vertices of $\overline{R}$; s_{j1} and s_{j2} are the beginning and end of Γ_j.** Under these assumptions, the solution of problem (1) satisfies $u \in C_{2m+2,\lambda}(\overline{R})$ (see ([7])). Let a/h and b/h be integers, $h \leqslant \min\{a, b\}/(2m + 2)$. Construct a grid of the straight lines $x, y = 0, h, 2h, \ldots$ and denote by R_h the set of grid nodes belonging to R, and by Γ_{jh} the set of nodes lying on Γ_j. Introduce the averaging operator A

$$Au(x,y) = (u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h))/4$$

and consider the recurrent sequence of systems of difference equations

$$u^q = \varphi_j \quad \text{on } \Gamma_{jh}, \quad j = 1,2,3,4, \tag{5}$$

$$u^q = Au^q - h^2 f/4 - D^{(m)} u^{q-1} \quad \text{on } R_h, \tag{4}$$

where $q = 1, 2, \dots, m+1$; $u^0 \equiv 0$; $D^{(m)}$ is a linear difference operator of order $2m+2$, which, upon substituting the functions u^q and u^{q-1} into equations (4), gives residuals of order $h^{2m+2+\lambda}$. The operator $D^{(m)}$, at all nodes where possible, is expressed in terms of operators for computing central differences, while at the remaining nodes the operator $D^{(m)}$ is constructed using Newton' s interpolation formula (see ([2])).

* Definition of the class $C_{k,\lambda}$ see, for example, in ([6]).

** If (2) holds but (3) is not fulfilled, then from the solution of problem (1) one subtracts

a certain elementary function, and for the remaining term conditions of the form (2) and (3) are satisfied (see ([3])).

Theorem 1. If (2)–(3) hold, then the solutions u^q, $q = 1, 2, \dots, m+1$, of the systems of difference equations (4)–(5) are representable on R_h in the form

$$u^q = u + \sum_{k=q}^{m} h^{2k} w_k^q + h^{2m+\lambda} r^q, \tag{6}$$

where u is the solution of problem (1); w_k^q are the traces on R_h of certain functions $w_k^q(x,y) \in C_{2m+2-2k,\lambda}(\overline{R})$, independent of h, such that

$$\|w_k^q\|_{C_{2m+2-2k,\lambda}(\overline{R})} \le c_q \left(\sum_{j=1}^{4} \|\varphi_j\|_{C_{2m+2,\lambda}(\Gamma_j)} + \|f\|_{C_{2m,\lambda}(\overline{R})} \right)^* ;$$

r^q are certain functions defined on R_h such that

$$\max_{R_h} |r^q| \le c_q \left(\sum_{j=1}^{4} \|\varphi_j\|_{C_{2m+2,\lambda}(\Gamma_j)} + \|f\|_{C_{2m,\lambda}(\overline{R})} \right).$$

Theorem 1 establishes that, in the case of a rectangle, the irregular component of the error of the approximate solution u^q of problem (1) has order $h^{2m+\lambda}$ for all $q \leq m+1$. The author supposes that the estimate given for the irregular error cannot be improved in order with respect to h. Theorem 1 refines and improves, for the Dirichlet problem, theorem 1 of ([3]), according to which the irregular error in u^q has order h^{2m+1-q}.

2. Let the boundary-value problem

$$\Delta u = f \quad \text{on } \Omega; \qquad u = \varphi \quad \text{on } \gamma, \tag{7}$$

be given, where γ is the boundary of the domain $\Omega \subset R$. Denote by: Ω_h the set of mesh nodes belonging to Ω, and such that their four neighboring nodes also belong to Ω; γ_h the set of the remaining nodes belonging to $\overline{\Omega}$. Introduce on the axis τ the operator I_δ^n

$$I_\delta^n \psi(0) \equiv \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{(n-k)!\,k!} \frac{\delta}{\delta+k} \psi(-k\overline{h})$$

and denote:

$$\lambda_n = \prod_{k=1}^{n} \frac{k}{\delta+k};$$

δ_n is the positive root of the equation

$$\sum_{k=1}^{n} \frac{n!}{(n-k)!\,k!} \frac{\delta}{\delta+k} = \frac{1}{2}.$$

Suppose that $\gamma \in {}_{2m+2}(B,\lambda)$ (see ([8])), $m \geq 1$, and h is so small that it is possible to write the following recurrent sequence of systems of difference equations:

$$\begin{aligned} u^q &= Au^q - h^2 f/4 - D^{(m)} u^{q-1} && \text{on } \Omega_h, \\ u^q &= I_\delta^{2m} u^q + \lambda_{2m}\varphi && \text{on } \gamma_h, \end{aligned} \tag{8}$$

where the operator I_δ^{2m} is constructed with step $\overline{h} = ([1/\delta_{2m}]+1)h$; the origin of coordinates on the axis τ is at the corresponding point γ_h; δh is the distance from the origin of coordinates to the point of intersection of the axis τ with γ, at which the value of φ is taken; $0 \leq \delta \leq \delta_{2m}$; the segment of the axis τ $[\delta h, -2m\overline{h}] \subset \Omega$; $u^0 \equiv 0$.

Theorem 2. If $\gamma \in {}_{2m+2}(B,\lambda)$, $\varphi \in C_{2m+2,\lambda}(\gamma)$, $f \in C_{2m,\lambda}(\overline{\Omega})$, $m \geq 1$, $0 < \lambda < 1$, then the solutions u^q, $q = 1, 2, \ldots, m+1$, of the systems of difference equations (8) are representable on $\Omega_h \cup \gamma_h$ in the form

Fig. 1

Figure 1: Fig. 1

$$u^q = u + \sum_{k=q}^{m} h^{2k} w_k^q + h^{2m+\lambda} \left(\ln \frac{a+b}{h} \right)^{q-1} r^q, \tag{9}$$

* Here and below, c_p, $p = 1, 2, ...$, denote constants independent of the factor standing to the right and of h.

where u is the solution of problem (7), w_k^q are the traces on $\Omega_h \cup \gamma_h$ of certain functions, independent of h, $w_k^q(x, y) \in C_{2m+2-2k,\lambda}(\overline{\Omega})$, such that

$$\|w_k^q\|_{C_{2m+2-2k,\lambda}(\overline{\Omega})} \le c_q(\|\varphi\|_{C_{2m+2,\lambda}(\gamma)} + \|f\|_{C_{2m,\lambda}(\overline{\Omega})});$$

r^q are certain functions defined on $\Omega_h \cup \gamma_h$ such that

$$\max_{\Omega_h \cup \gamma_h} |r^q| \le c_q(\|\varphi\|_{C_{2m+2,\lambda}(\gamma)} + \|f\|_{C_{2m,\lambda}(\overline{\Omega})}).$$

Theorem 2 substantially improves the result of ([1], [2]), since in (9) the irregular error contains the inessential factor

$$\left(\ln \frac{a+b}{h} \right)^{q-1}$$

instead of the quantity h^{2-2q} entering into the corresponding estimate given in ([1], [2]).

Fig. 1

3. In ([4], [5]) the recurrent system of difference equations is considered

$$\begin{aligned}
&\frac{1}{h^2}\Delta_h u_0 = f, && \delta_h u_0 = \varphi, \\
&\frac{1}{h^2}\Delta_h u_1 + L_1(u_0) = 0, && \delta_h u_1 = \delta_1(u_0), \\
&\dots\dots\dots\dots\dots\dots && \\
&\frac{1}{h^2}\Delta_h u_{q-1} + L_1(u_{q-2}) + \dots + L_{q-1}(u_0) = 0, && \delta_h u_{q-1} = \delta_1(u_{q-2}) + \dots + \delta_{q-1}(u_0),
\end{aligned} \tag{10}$$

where Δ_h, L_k are certain difference operators defined on Ω_h;

$\Delta_h = 4(A - E)$; $Eu \equiv u$; δ_h is an operator defined on γ_h such that, for a sufficiently smooth solution of problem (7),

$$\delta_h u = \varphi + h^2\delta_1(u) + \dots + h^{2q-2}\delta_{q-1}(u) + O(h^{2q}); \tag{11}$$

$\delta_1, \delta_2, \dots, \delta_{q-1}$ are certain difference operators.

In ([4], [5]) the following theorem is formulated (in the author' s notation). If $\gamma \in \Pi_{4q}(B, \lambda)$, $\varphi \in C_{4q,\lambda}(\gamma)$, $f \in C_{4q-2,\lambda}(\overline{\Omega})$, then the solution u of problem (7) is representable on $\Omega_h \cup \gamma_h$ in the form of the asymptotic series

$$u = u_0 + h^2 u_1 + \dots + h^{2q-2}u_{q-1} + O(h^{2q}), \tag{12}$$

where $u_0, u_1, \dots, u_{q-1}$ are the solutions of the system of difference equations (10).

The incorrectness of the proof in ([5]) of this theorem was noted in ([9]). We give an example establishing the erroneousness of the theorem. Let γ contain the rectilinear segment KM, intersecting at a small angle $\theta > 0$ the vertical line of the grid at the point T, located midway between the nodes 4 and 9 (Fig. 1). This arrangement of the points $1, 2, \dots, 12$ takes place if $a = 2b$; T lies at the center R; $h = b/(2k + 1)$, where k is any integer exceeding a certain constant. Let also $\gamma \in \Pi_8(B, \lambda)$, $f = 4$, and the function φ be such that $u = x^2 + y^2$ in Ω. According to (11), take the following expressions for $\delta_h u$ and $\delta_1(u)$ at point 4:

$$\delta_h u = u^4(1 + \delta) - u^5\delta, \tag{13}$$

$$\delta_1(u) = \delta(-u^4(5+6\delta+\delta^2)/6+u^5(4+5\delta+\delta^2)/2-u^6(3+4\delta+\delta^2)/2+u^7(2+3\delta+\delta^2)/6)/h^2, \tag{14}$$

where δ is a parameter such that δh is equal to the distance between the points 4 and $4'$; u^k is the value of u at the k-th point. The function $w = u_0 - u$ satisfies the system of difference equations

$$\Delta_h w = 0, \qquad \delta_h w = \delta(1 + \delta)h^2, \tag{15}$$

where δ is the parameter on γ_h (see (13)). Let $0 \le \delta < 1$. Then

$$0 \le w < 2h^2 \quad \text{on } \Omega_h \cup \gamma_h. \tag{16}$$

Since at point 4 $\delta = \varepsilon$, and at point 10 $\delta = 1 - \varepsilon$, where $\varepsilon = (\operatorname{tg}\theta)/2$, it follows, according to (13), (15), (16), that

$$w^4 < \varepsilon \cdot 3h^2, \tag{17}$$

$$w^{10} = (1-\varepsilon)h^2 + w^{11}(1-\varepsilon)/(2-\varepsilon) * . \qquad (18)$$

Taking (15)–(18) into account, it is not difficult to calculate that for $\varepsilon \le 10^{-3}$ at point 4 $\delta_1(w) > \varepsilon/20$. Let $\overline{w} = u_0 + h^2 u_1 - u$. Since $\delta_h \overline{w} = h^2 \delta_1(w)$, it follows from what has been said and from (13) that for $\varepsilon = \varepsilon_0 = 10^{-3}$ $\max\{\overline{w}^4,\ |\overline{w}^5|\} > \varepsilon_0 h^2/50$, which contradicts the theorem cited above ([4], [5]), according to which $\overline{w} = O(h^4)$.

V. S. Korolyuk pursued the aim of simplifying the method of refinement by differences of higher orders by using in the system of difference equations (10) the simple operator δ_h (instead of I_δ^{2m}) and by introducing, both on Ω_h and on γ_h, corrections through the preceding solution by means of difference operators whose accuracy increases gradually from refinement to refinement, and is not fixed in advance. But, as the example shows, such a scheme as a whole, generally speaking, does not lead to the desired result. V. S. Korolyuk' s proposal on the gradual complication of the difference operators calculating the corrections can possibly be realized only at interior nodes, and even then, apparently, not at all of them. At nodes close to γ_h, where operators calculating central differences cannot be applied, it will evidently be necessary to use operators of order $2m$, and these operators will depend on q.

A detailed analysis of the question is being published in ([10]).

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
24 II 1965

CITED LITERATURE

[1] E. A. Volkov, DAN, 96, No. 4 (1954).
[2] E. A. Volkov, Collection "Computational Mathematics," Publishing House of the Academy of Sciences of the USSR, No. 4, 1957, p. 62.
[3] E. A. Volkov, DAN, 150, No. 3 (1963).
[4] V. S. Korolyuk, DAN, 101, No. 6 (1955).
[5] V. S. Korolyuk, Ukrainian Mathematical Journal, 7, No. 4 (1955).
[6] O. A. Ladyzhenskaya, N. N. Ural' tseva, UMN, 16, No. 1 (1961).
[7] E. A. Volkov, DAN, 147, No. 1 (1962).
[8] N. M. Gunter, *Potential Theory and Its Application to the Basic Problems of Mathematical Physics*, 1953.
[9] RZh Matematika, 1956, ref. 9099.
[10] E. A. Volkov, *Differential Equations*, 1, No. 7 (1965); 1, No. 8 (1965).

* From (16)–(18), for $\varepsilon \le 10^{-1}$, in particular, it follows that Lemma 2 ([4]) and Theorem 2 ([5]) are incorrect.